#1: LaTeX and Set Theory

September 13, 2008

1 LaTeX

The first part of your assignment this week is to become familiar with LaTeX, which is the primary system we will be using to communicate. The accompanying LaTeX tutorial will guide you through the process of installing it on your computer, and show you the basics of how to prepare a document in LaTeX format.

2 Set Theory

These marginal notes name the topic being discussed to their right; they are to help you find things when you refer back to these notes later.

The second part of your assignment is a basic introduction to set theory. You should make a copy of `solution-template.tex`, rename it to something like `01-set-theory-solutions.tex`, and fill it in with your solutions, using the sample solution (`solution-example.tex`) and the LaTeX tutorial as guides. Be sure to also read the expectations outlined in the syllabus. When you are done, you should turn in the `.tex` file with your solutions, along with the generated PDF file.

2.1 Definitions

set

> A *set* is a collection of objects, which are called *elements*. The order of the elements does not matter, and each element may occur no more than once.

set examples

For example, $\{1, 2, 5\}$ denotes a set with three elements: 1, 2, and 5. $\{2, 5, 1\}$ is the same set, since the order of the elements does not matter. $\{2, 2, 2\}$ is not a valid set, because the element 2 occurs more than once. Note that the elements of a set do not have to be numbers; they could be any sort of object, like people, types of cheese, triangles, binary operations, or even other sets.

Problem 1. Which of the following are valid sets?

(a) $\{5, 4, 3, 1\}$

(b) $\{2, 5, 7, 2\}$

(c) $\{\{2, 5, 1\}, \{2, 5\}, \{2\}\}$

(d) $\{\{\{\pi\}\}\}$

2.2 Notation

set notation

As seen above, one way to describe a set is to literally list its elements and place them in curly braces, like this: $\{1, 3, 69\}$. (Remember that because curly braces have special meaning to LaTeX, you will have to put backslashes in front of them.)

special sets: $\emptyset$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

There are also some important sets which have special notation. Here are a few:

- $\emptyset$ denotes the empty set—the unique set which contains no elements. Sometimes it is also written $\{\}$.
- $\mathbb{N}$ stands for the set of all natural numbers, that is, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
- $\mathbb{Z}$ stands for the set of all integers, that is, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of all rational numbers, all real numbers, and all complex numbers, respectively.

Problem 2. Are $\emptyset$ and $\{\emptyset\}$ the same? If not, what is the difference?

Problem 3. Let S_0 denote the empty set, and define S_n (for any positive integer n) to be the set $\{S_0, S_1, \dots, S_{n-1}\}$. For example, $S_1 = \{S_0\} = \{\emptyset\}$, and $S_2 = \{S_0, S_1\} = \{\emptyset, \{\emptyset\}\}$. Write out S_4.[1]

$\in$, "element of"

To specify that something is an element of a particular set, use the $\in$ symbol. For example, $a \in \mathbb{Z}$ means that a is an integer. $y \in \{0, 1\}$ means that y is either zero or one. It is common to write several things in front of $\in$ separated by commas; for example, $a, b, c \in \mathbb{Z}$ means that a, b, and c are all integers. To say that something is *not* an element of a set, you can use the symbol $\notin$, just like you use $\neq$ to indicate that two things are not equal.

[1] Believe it or not, this is actually a common way to *formally* define the natural numbers using nothing other than set theory!

Problem 4. Which of the following statements are true, and which are false?

(a) $2 \in \{2, 5, 7\}$

(b) $3 \notin \mathbb{Z}$

(c) $9.4 \notin \mathbb{Z}$

(d) $\{2\} \in \{2, 5, 7\}$

(e) $3 \in \{\{1\}, \{2\}, \{3\}\}$

(f) $\{2, 5\} \notin \{\{6, 7\}, \{2, 5, 1\}\}$

(g) $\emptyset \in \{5, \varphi, \emptyset\}$

such-that notation

Another way to describe a set is using so-called "such-that" notation. The notation

$$\{P \mid Q\},$$

read "P such that Q," denotes the set of all values of P for which Q is true. For example,

$$\{x \mid x \in \mathbb{Q} \text{ and } x > 6\}$$

denotes the set of all rational numbers greater than 6. As another example,

$$\{x^2 \mid x \in \mathbb{Z}\}$$

denotes the set of all perfect squares. As a third example, $\mathbb{Q}$ can be defined this way:

$$\mathbb{Q} = \{p/q \mid p, q \in Z \text{ and } q \neq 0\}.$$

interval notation

Yet another way to describe certain special sets is *interval* notation. It's possible you've seen this notation before. In particular,

$$[a, b]$$

denotes the set of all *real* numbers between a and b (inclusive). In other words,

$$[a, b] = \{x \mid x \in \mathbb{R}, x \geq a, x \leq b\}.$$

(Note that using commas as above is a common shorthand for "and," *i.e.* we write '$x \in \mathbb{R}, x \geq a$' instead of '$x \in \mathbb{R}$ and $x \geq a$,' and so on.) The square brackets around $[a, b]$ indicate that both endpoints are included in the set; parentheses indicate that one or both endpoints are not included. For example, $(3, 5]$ is the set of all real numbers which are *greater than* (but not equal to) 3, and less than or equal to 5. Likewise, $[3, 5)$ indicates that 3 is included but not 5; $(3, 5)$ indicates that neither endpoint is included.

If one end of an interval has no endpoint, we write $-\infty$ or ∞ enclosed in a parenthesis. For example, $(-\infty, 6]$ is the set of all real numbers less than or equal to 6.

2.3 Set operations

There are a few fundamental operations we can perform on sets.

union, $\cup$

The *union* of two sets, denoted $S \cup T$, is the set which contains all the elements of S and all the elements of T. For example, $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$. Notice that it is OK for S and T to overlap; their union only contains one copy of each element, even if it occurs in both S and T.

intersection, $\cap$

The *intersection* of two sets, denoted $S \cap T$, is the set which contains all the elements which are in *both* S and T. For example, $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$.

difference, $\setminus$

The *difference* of two sets, denoted $S \setminus T$ (also sometimes $S - T$), is the set which contains all the elements which occur in S but not in T. For example, $\{1, 2, 3\} \setminus \{3, 4, 5\} = \{1, 2\}$.

complement, $\overline{S}$

The *complement* of a set, denoted $\overline{S}$, is the set which contains everything which is *not* an element of S. This only makes sense with respect to some "universal set" of all objects under consideration (sometimes called the "universe of discourse"), which is usually clear from context. For example, it's nonsensical (or at least, not very useful) to say that $\overline{\{2\}}$ is the set which contains "everything except 2", including 6, 97.3, Archimedes, the Leaning Tower of Pisa, dirty socks... Rather, we would say that, with respect to the integers, for example, $\overline{\{2\}}$ contains all integers except 2; with respect to the universal set $\{1, 2, 3\}$, the complement of $\{2\}$ is $\{1, 3\}$, and so on. Another way to say this is that if U is the universal set, then $\overline{S} = U \setminus S$.

cardinality, $|S|$

Finally, the *cardinality*, or size, of a set is simply the number of elements it contains. The cardinality of S is usually denoted $|S|$ (or sometimes $\#S$). For example, $|\{2, 4, 6, 7\}| = 4$. Of course, this definition breaks down when we start talking about infinite sets—for example, what is $|\mathbb{Z}|$? We'll explore the cardinality of infinite sets in a few weeks.

Problem 5. (a) What is $[1, 5) \cap [2, 6)$?

(b) Is $2 \in (-\infty, 0) \cup [1/2, 3) \cup (9, 12]$?

(c) What is $\overline{(-\infty, 0) \cup [1/2, 3) \cup (9, 12]}$?

For problems 6–7, assume that the following sets are defined. All set complement operations are with respect to the universal set $\mathbb{Q}$.

$$\begin{aligned}
A &= \{1, 2, 3\} \\
B &= \{0, 5, 3, 8, 1\} \\
C &= \{1, 7, 9, 3\} \\
D &= \overline{\{1/2, 3/4, 5/7\}} \\
E &= \{x^2 \mid x \in \mathbb{Z}\} \\
F &= \{p/(2^q) \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\} \\
G &= \{|A|, |B|, |C|\}
\end{aligned}$$

Problem 6. List the elements of each of the following sets, or describe them if there are an infinite number of elements.

(a) $A \cup B$

(b) $A \cap C$

(c) $\overline{D} \cap F$

(d) $A \cup (E \cap C)$

(e) $B \setminus G$

Problem 7. For each set, write down an expression which is equal to it, using only the sets A through G defined above and the set operations.

(a) $\{3, 1, 9, 7, 5, 4\}$

(b) $\{1\}$

(c) $\{5/7\}$

(d) $\emptyset$

Problem 8. Explain why, for any sets A and B, it is always true that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Problem 9. Is it also always true that $\overline{A \cap B} = \overline{A} \cup \overline{B}$? If so, explain why; if not, give a counterexample (particular sets A and B for which it is not true).

Problem 10. Which set has more elements: the set of all integers, or the set of all even integers? Does it even make sense to ask this question? Justify your answers. Note that I don't care if you get this question "right" (we'll talk about the "right" answer in a few weeks) but just for you to think about it carefully.

3 Set notation and LaTeX

You can, of course, look at the LaTeX file for this assignment to see commands used to produce special set-related symbols. However, for your convenience I have provided Table 1 as a reference.

Symbol	command
$\{$	`\{`
$\mid$	`\mid`, `\suchthat`
$\}$	`\}`
$\in$	`\in`
$\not\in$	`\not \in`
$\emptyset$	`\emptyset`
$\mathbb{N}$	`\N`
$\mathbb{Z}$	`\Z`
$\mathbb{Q}$	`\Q`
$\mathbb{R}$	`\R`
$\mathbb{C}$	`\C`
$\cup$	`\cup`, `\union`
$\cap$	`\cap`, `\intersect`
$\setminus$	`\setminus`
$\overline{S}$	`\overline{S}`
∞	`\infty`

Table 1: LaTeX commands for set-related symbols.

`\mid`, `\cup` and `\cap` are standard LaTeX commands, for which I have provided the easier-to-remember aliases `\suchthat`, `\union` and `\intersect`, respectively, in `precalc.sty`. Feel free to use either, or to define your own aliases using `\newcommand`. I have also provided `\N` and company as aliases for `\mathbb{N}` and so on, which typesets the characters using the special double-stroked "blackboard bold" font which is commonly used for these symbols.

