

#8: Fundamentals of Trigonometry

October 30, 2008

The word “trigonometry” literally means “triangle measurement”: historically, trigonometry is indeed rooted in the study of triangles, and you probably remember seeing a bit of it in geometry class. However, since ancient Greece it has come to have much wider scope and connections to many other areas of mathematics.

1 Radians

Have you ever wondered exactly why we use *degrees* to measure angles? Degrees are pretty arbitrary, when you stop to think about it. Why are there *three hundred sixty* degrees in a full circle? Why not five hundred, or one hundred? Or why not just say “1/4 of a revolution” instead of “90 degrees”?

As best we can tell, the practice of using degrees to measure angles comes to us from the ancient Babylonians, via the ancient Greeks. The Babylonians were avid astronomers, and noticed that the stars seemed to rotate about 1/360 of a full rotation from night to night, so decided to use that as the basis for measuring angles and rotations. It probably also helped that 360 has many divisors, so it was easy to work with various fractions of 360.

Problem 1. How many different positive divisors (including 1 and 360 itself) does 360 have? List them. For example, 12 has 6 divisors: 1, 2, 3, 4, 6, and 12.

So, the fact that there are 360 degrees in a circle actually comes from the fact that there are (about) 360 days in a year.

Well, that’s nice, but it’s still rather arbitrary, and a bit too concrete for our abstract modern tastes. Can you imagine if I told you that from now on, we will use a picture of two hands holding apples instead of the + symbol, because when we add things it is like putting two apples together? Well, that’s essentially what the system of degrees is like, if you think about it: we are basing our definition on some arbitrarily chosen aspect of the real world, rather than using a more elegant, abstract definition.

So, what's a less arbitrary way to measure angles? Well, here's an idea: if we think of an angle as a sector of a unit circle (by a "unit circle" I mean a circle with a radius of 1), we can measure the angle by the *length* of the circular arc that it corresponds to. In other words, to see how many radians a certain angle corresponds to, draw the angle on a circle of radius 1, and measure the part of the circle's circumference which is inside the angle. Figure 1 illustrates this idea.

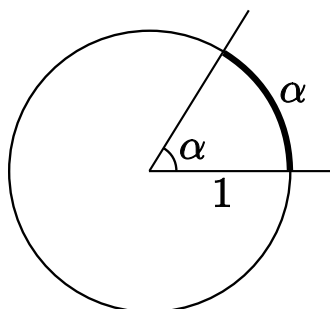


Figure 1: Measuring angles by distance on a unit circle.

So, in the picture, we represent angle α (which could be any angle) by the length of the bold arc; the 1 reminds us that we are using a unit circle (a circle with radius 1). These new units for measuring angles are known as *radians*.¹

Problem 2. What is the circumference of a unit circle?

Problem 3. How many radians does 360° correspond to? Give your answer in exact form.

Problem 4. Convert from degrees to radians. Leave your answers in exact form.

- (a) 180°
- (b) 90°
- (c) 60°
- (d) 45°
- (e) 30°

¹Don't ask, I have no idea why.

Problem 5. Convert from radians to degrees.

(a) 4π

(b) $\pi/7$

Problem 6. In general, what is the formula for converting from degrees to radians? From radians to degrees?

2 Angles

We say that an angle is in *standard position* if its vertex is at the origin, and it is measured counterclockwise from the x -axis. Since angles can be in any position and orientation and still be the same angle, it simplifies things if we always assume angles are in standard position. As you may recall from geometry, an angle is formed from two rays with a common vertex; so another way to describe an angle in standard position is any angle which has the positive x -axis as one of its rays. The other ray is called the *terminal ray*. Figure 2 shows an angle in standard position.

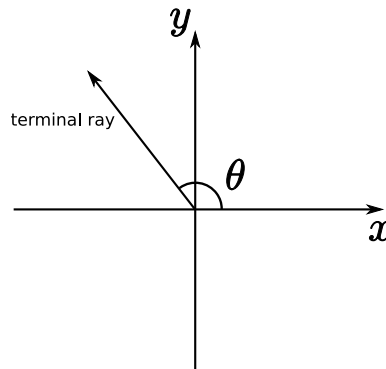


Figure 2: An angle θ in standard position.

Unlike what you may have learned in geometry, from now on it is perfectly acceptable to have angles bigger than 2π radians, or even negative angles. Angles bigger than 2π radians correspond to going around the circle multiple times; negative angles are measured *clockwise* starting from the x -axis. Figure 3 shows some examples. The red angle is a negative angle; the blue angle is larger than 2π radians.

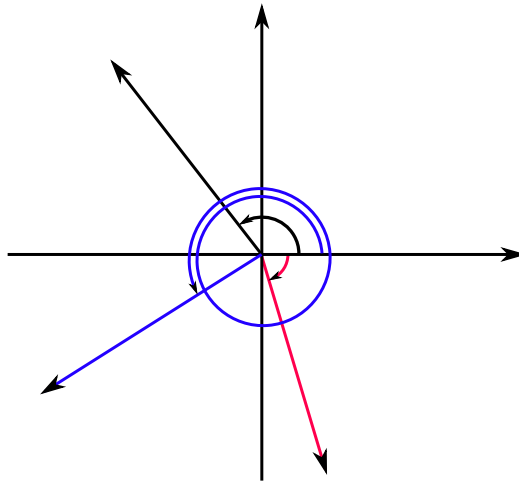


Figure 3: Some example angles.

If two angles “end up” in the same place on a circle, we say the angles are *coterminal*, since their *terminal* rays *coincide*. For example, the three angles shown in Figure 4 are all coterminal. Even though they are different angles, in some sense they are all equivalent.

Problem 7. For each angle, find another angle $\alpha \in [0, 2\pi)$ which is coterminal with the given angle.

- (a) 3π
- (b) $95\pi/6$
- (c) $-3\pi/2$

3 Sine and cosine

The sine and cosine functions are fascinating and foundational—they show up all over the place in many different areas of mathematics, often when you least expect them! They also have many interesting properties that make them fun to play with in their own right.

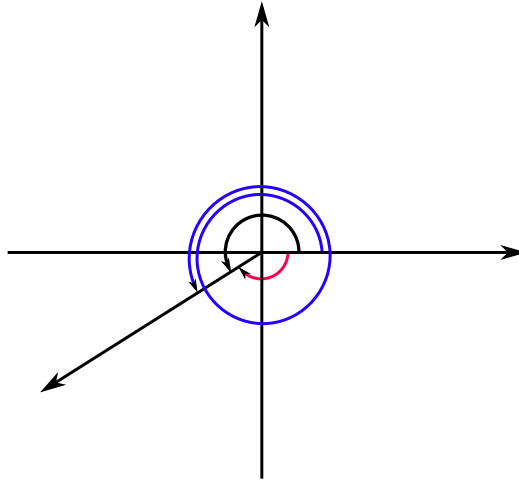


Figure 4: Three coterminal angles.

3.1 Sine and cosine in right triangles

You have probably already learned about sine and cosine in the context of right triangles. If θ is one of the angles in a right triangle (other than the right angle itself), then $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, that is, the ratio between the length of the opposite leg of the triangle and the length of the hypotenuse.² Likewise, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, the ratio between the lengths of the adjacent leg and the hypotenuse. See Figure 5.

You also may have seen the mnemonic device “SOHCAHTOA” to help you remember these definitions: Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse (and Tangent is Opposite over Adjacent).

These right-triangle based definitions are important in several ways:

1. They represent the historical roots of the sine and cosine functions. This is how sine and cosine were first defined.
2. They are useful in applying the tools of trigonometry to real-world problems involving right triangles.
3. They provide useful intuition when dealing with generalized definitions of sine and cosine.

²Even though we abbreviate the sine function as sin, you should still pronounce it “sine”. The same goes for cosine.

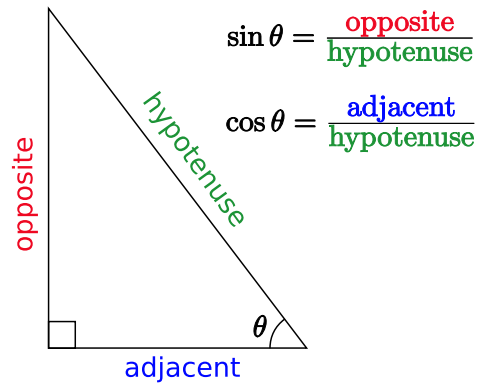


Figure 5: Sine and cosine for a right triangle

Problem 8. What is $\sin \pi/4$? Hint: draw an appropriate triangle.

Problem 9. Triangle ABC is a right triangle with B as the right angle. Side AB is 8 units long, and side BC is 3 units long. What is $\cos A$? As always, give your answer in exact form.

As hinted previously, the right triangle definitions of sine and cosine are not general enough. In particular, they only define sine and cosine for angles between 0 and $\pi/2$; we would like a definition of sine and cosine that works for any angle.

3.2 General definitions

Suppose we have some angle θ . Let's draw θ in standard position, on top of a unit circle.

The terminal ray of θ intersects the unit circle at some point; suppose that point has coordinates (x, y) , as shown in Figure 6. Then we define

$$\cos \theta = x \tag{1}$$

and

$$\sin \theta = y. \tag{2}$$

What could be simpler? Notice that this definition works for *any* angle: we just draw the angle in standard position, see where its terminal ray intersects

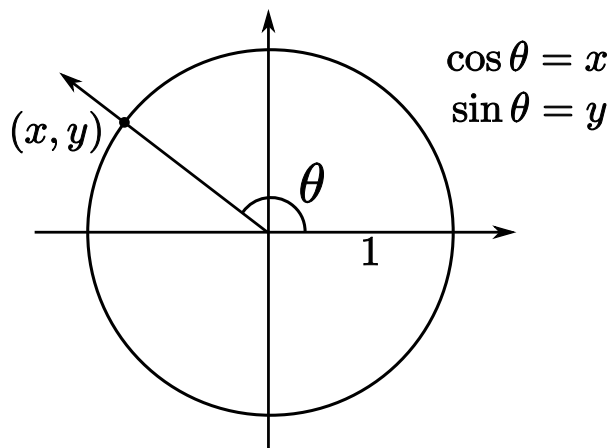


Figure 6: Defining sine and cosine for any angle θ

the unit circle, and pick the x or y coordinate depending on whether we want the cosine or sine.

Problem 10. Invent a silly mnemonic to help you remember that cosine goes with x , and sine goes with y .

Problem 11. Evaluate.

- (a) $\sin \pi$
- (b) $\cos(\pi/2)$
- (c) $\sin(3\pi/2)$
- (d) $\cos \pi$
- (e) $\sin(99\pi)$
- (f) $\sin(13\pi/4)$

Problem 12. I claimed that the new definitions of sine and cosine are more general than the old right-triangle definitions—so they had better be the same for angles between 0 and $\pi/2$! You don't have to take my word for it. Let's see why they are the same.

Start by drawing a picture like Figure 6 for an angle θ between 0 and $\pi/2$. (You don't have to actually turn in the drawing; making the drawing will help you answer the following questions.) Call the point where θ 's terminal

ray intersects the unit circle P . As in Figure 6, suppose the coordinates of P are (x, y) .

- (a) What is the distance between the origin and P ? Why?
- (b) Draw a vertical segment from P to the x -axis, and call the point where it intersects the x -axis Q . What is the length of segment PQ ?
- (c) What is the distance from the origin to Q ?
- (d) What is $\sin \theta$ according to the right triangle definition? Why?
- (e) What is $\sin \theta$ according to the more general definition? Why?
- (f) Is $\cos \theta$ the same for the two definitions as well? Why or why not?

4 L^AT_EX Hints

Here are some L^AT_EX hints for this assignment:

- You can make a degree symbol with a superscript `\circ`. For example, `360^\circ` produces 360° .
- You can use the `\pi` command to make a π . In fact, you can make any Greek letter by using the command with the same name; use uppercase commands for uppercase versions of Greek letters (for example, `\gamma` makes a lowercase gamma, while `\Gamma` makes an uppercase one). Table 1 on page 9 lists all the Greek letters.

If there is a dash (—) in the uppercase column, it means there is no command for an uppercase version of that Greek letter, since it looks exactly like an uppercase Roman letter. For example, uppercase alpha just looks like A, so there is no need for a separate command. There is no `\omicron` command (omicron comes between xi and pi in the Greek alphabet) since it looks exactly like a lowercase o.

Note that if you want to express summation or product notation, you should use `\sum` or `\prod`, not `\Sigma` or `\Pi`.

- Whenever you write the sine or cosine functions in an equation, you should use the special commands `\sin` and `\cos`. If you just write `sin` or `cos` (without the backslash), L^AT_EX will think you mean s times i

Name	Lowercase	Uppercase
<code>\alpha</code>	α	—
<code>\beta</code>	β	—
<code>\gamma</code>	γ	Γ
<code>\delta</code>	δ	Δ
<code>\epsilon</code>	ϵ	—
<code>\zeta</code>	ζ	—
<code>\eta</code>	η	—
<code>\theta</code>	θ	Θ
<code>\iota</code>	ι	—
<code>\kappa</code>	κ	—
<code>\lambda</code>	λ	Λ
<code>\mu</code>	μ	—
<code>\nu</code>	ν	—
<code>\xi</code>	ξ	Ξ
<code>\pi</code>	π	Π
<code>\rho</code>	ρ	—
<code>\sigma</code>	σ	Σ
<code>\tau</code>	τ	—
<code>\upsilon</code>	υ	Υ
<code>\phi</code>	ϕ	Φ
<code>\chi</code>	χ	—
<code>\psi</code>	ψ	Ψ
<code>\omega</code>	ω	Ω

Table 1: Greek letters

times *n*, or *c* times *o* times *s*, and typeset them in italics with no space afterwards. See the difference?

Wrong: *sin* θ + *cos* θ Right: $\sin \theta + \cos \theta$