#9: Fundamentals of Trigonometry, Part II

November 1, 2008

In the last assignment, you learned general definitions of the sine and cosine functions. This week, we will explore some of the special properties of these functions (there are many)!

do not panic. WARNING: this is the point at which many people decide that "trig sucks." The reason is that they think (or are taught) that the way to learn trig is to memorize a huge number of equations, formulas, and procedures. But that is exactly the *wrong* way to learn trigonometry! What's really cool about trigonometry is how many interesting equations, formulas, and procedures *can be derived* from just a few fundamental definitions, and how many interesting interrelationships there are between all of these. The only things you need to memorize are those few fundamental definitions, which gives you a basis from which to understand (and enjoy) all the ramifications, without stressing about memorizing a gazillion things.

1 Fundamental fundamentals

So, what are those fundamental definitions? There are only two; you learned them both last week. Here they are, repeated for your convenience:

> There are 2π radians in a full revolution.
>
> and
>
> Draw θ as an angle in standard position, on top of a *unit* (radius 1) circle. The point where θ's terminal ray intersects the unit circle has coordinates $(\cos\theta, \sin\theta)$. (See Figure 1.)

That's it! That's *all you need to know.* If you ever get confused, you can just come back to these two definitions.
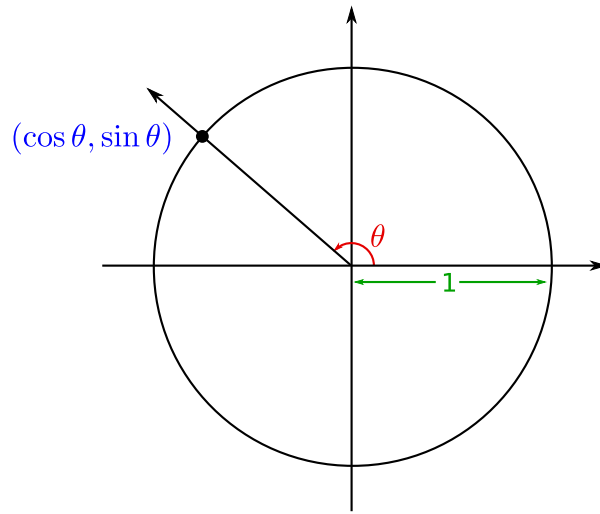
Figure 1: Definition of sine and cosine

Problem 1. Write out the two fundamental fundamentals (complete with diagram for the second) on a separate sheet of paper, *without peeking*! After writing them out, compare and see how you did. If you got confused or forgot something, write them out again, repeating until you can confidently write them down from memory and are sure you understand them.

I am actually serious about this! For your answer to Problem 1, you should submit the statement "I fully followed the instructions in this problem, cross my heart and may my toes be eaten by ravenous tarantulas if I am not telling the truth." Of course, you should only write it as your answer if it is a true statement. (Because otherwise your toes will be eaten by ravenous tarantulas.)

2 Special angles

The definition of sine and cosine is very useful for *reasoning* about these functions, but in general it is not very useful for actually *computing* the sine or cosine of a particular angle. (You would have to draw the angle using a protractor—not a very precise operation—and then measure some distances in your diagram with a ruler—also not very precise.)

In general, if you want to know the sine or cosine of any old angle, you

should use a calculator. [1] [2] [3] However, there are certain special angles for which we know the *exact* sine and cosine, and it's very important to know them (they tend to come up a lot—after all, they are special!).

Problem 2. For each part, give your answer in exact form and *explain* the reasoning behind your answer. Use the fundamental fundamentals!

    (a) What is $\sin 0$?

    (b) What is $\sin(\pi/2)$?

    (c) What is $\cos 0$?

    (d) What is $\cos(\pi/2)$?

Problem 3. Consider the diagram in Figure 2. It shows an angle of $\pi/4$ (otherwise known as 45°) drawn on a unit circle.

    (a) What is special about triangle OPQ? (*Hint*: the angles of any triangle add up to π...)

    (b) What is the length of segment OP?

    (c) What is the length of segment OQ? (*Hint*: there's a Theorem named after some dead Greek guy that may be able to help you...)

    (d) What is the length of segment PQ?

    (e) What can you conclude about $\sin(\pi/4)$? Give your answer in exact form, and check it using a calculator.

    (f) What can you conclude about $\cos(\pi/4)$?

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[1]This is a good time to mention that you should always be very careful about which mode your graphing calculator is in—degree mode or radian mode. When it is in degree mode, it expects angles in degrees; when in radian mode, it expects angles in radians. Obviously, if you type in an angle in radians, but your calculator thinks it is in degrees (or vice versa), you will get very strange (and wrong) answers.

[2]Before calculators were invented, people (very tediously) made huge books full of the sine and cosine of any angle (to five or six or more decimal places), so if you wanted to know the sine or cosine of an angle, you could just look up the angle in the book. Handy!

[3]There are better ways to calculate the sine and cosine of an angle; in particular, your calculator does not actually calculate sine and cosine using this definition, by drawing a really accurate picture and so on. Both sine and cosine have elegant *infinite series expansions* that allow you to calculate the sine or cosine of an angle with as much accuracy as you want. But to understand these methods you'll have to wait until calculus!
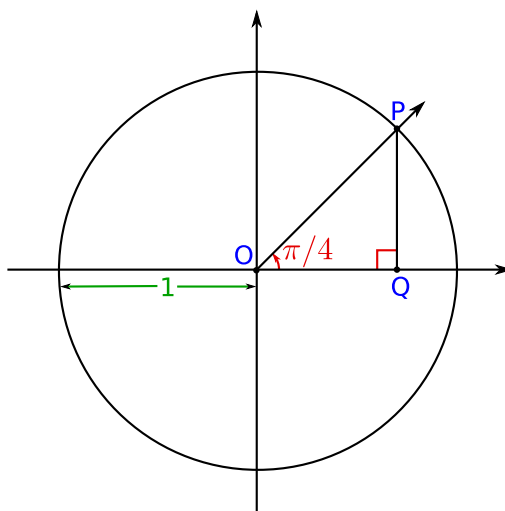
Figure 2: Determining $\sin(\pi/4)$ and $\cos(\pi/4)$

Problem 4. Now take a look at the diagram in Figure 3. It shows an equilateral triangle ABC with sides of lenth 2. As you may remember from geometry, an equilateral triangle is one which has three equal sides; all of the angles in an equilateral triangle are also the same (in particular, they are all $\pi/3$ since the angles of any triangle add up to π).

Triangle ABC has been cut in half by altitude AD, with length h. Therefore segment DC has length 1 (half of segment BC) and $\angle DAC$ is $\pi/6$ (half of $\angle BAC$).

    (a) Find h.

    (b) What is $\sin(\pi/3)$?

    (c) What is $\cos(\pi/3)$?

    (d) What is $\sin(\pi/6)$?

    (e) What is $\cos(\pi/6)$?

Problem 5. Let's summarize what we've learned so far. Fill in Table 1 on page 5.
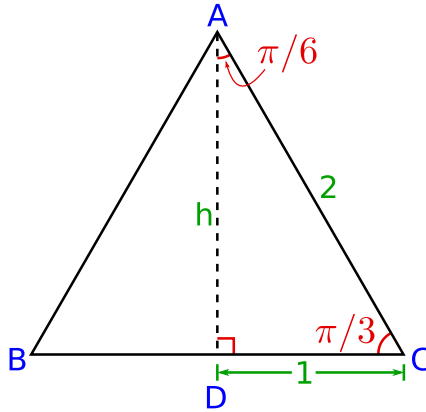
Figure 3: Determining the sine and cosine of $\pi/3$ and $\pi/6$

θ	$\sin\theta$	$\cos\theta$
0		
$\pi/6$		
$\pi/4$		
$\pi/3$		
$\pi/2$		

Table 1: Sine and cosine of special angles

3   Properties of sine and cosine

Problem 6. Using your calculator, evaluate the sine of $4\pi/5$, $14\pi/5$, $24\pi/5$, and $34\pi/5$. What do you notice? Can you explain why this happens?

Problem 7. Use a graphing calculator to look at graphs of $\sin x$ and $\cos x$. For best results, select the "Zoom Trig" zoom level.

(a) Both graphs appear to have a repeating pattern. Functions with a repeating pattern like this are called *periodic*. How often do the graphs repeat? Can you explain why?

(b) Notice that the graphs of $\sin x$ and $\cos x$ look very similar. What is the difference between them? (It may help to graph both functions at the same time.)

Problem 8. Consider Figure 4, which shows an angle θ and its negative both in standard position. Note that θ could be anything; Figure 4 just shows one particular example. The terminal rays of θ and $-\theta$ will always be mirror images of each other across the x-axis, no matter what θ is.
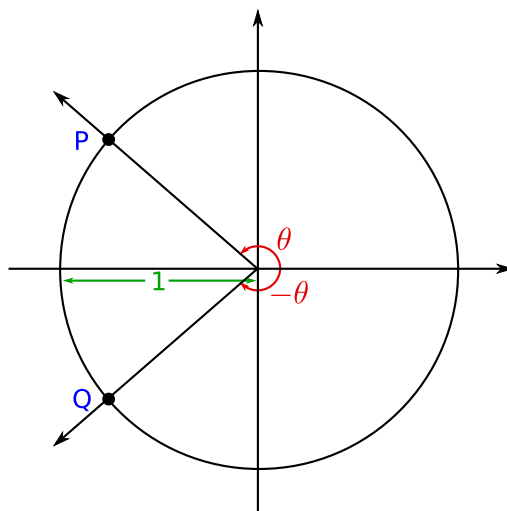

Figure 4: θ and $-\theta$

(a) How are the x-coordinates of P and Q related? What can you conclude about $\cos\theta$ and $\cos(-\theta)$?

(b) How are the y-coordinates of P and Q related? What can you conclude about $\sin\theta$ and $\sin(-\theta)$?

4 Other trigonometric functions

There are several other trigonometric functions in common use; however, they can all be defined in terms of sine and cosine. Tangent (tan) is sine divided by cosine; cosecant (csc) and secant (sec) are the reciprocals of sine and cosine, respectively; and cotangent (cot) is the reciprocal of tangent, that is, cosine divided by sine.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Problem 9. Evaluate. Give your answers in exact form.

(a) $\tan(\pi/4)$

(b) $\csc(\pi/2)$

(c) $\sec(\pi/6)$

(d) $\cot(\pi/3)$

(e) Try typing $\tan(\pi/2)$ into your calculator. What happens? Why?