#10: Reference angles

November 12, 2008

1 Sine and cosine, redux

special angles recap

Let's take a minute to remember the things we've learned about the sine and cosine functions. Last week, recall that you filled in a table which looked something like Table 1.

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

Table 1: All you will ever need to know!

Actually, it looked slightly different, but it was equivalent:

Problem 1. Show that $\sqrt{1/2} = \sqrt{2}/2$. (*Hint*: multiply by a something suitably equal to 1...)

special angles, more specialler

Table 1 shows the way that this table is most commonly presented. However, there is another way to write the same table which makes it easier to remember: see Table 2 (you should convince yourself that this is the same).

θ	$\sin\theta$	$\cos\theta$
0	$\sqrt{0/4}$	$\sqrt{4/4}$
$\pi/6$	$\sqrt{1/4}$	$\sqrt{3/4}$
$\pi/4$	$\sqrt{2/4}$	$\sqrt{2/4}$
$\pi/3$	$\sqrt{3/4}$	$\sqrt{1/4}$
$\pi/2$	$\sqrt{4/4}$	$\sqrt{0/4}$

Table 2: All you will ever need to know, now even easier to remember!

special angles, the movie

Nifty, eh? And here's a nice diagram showing the same thing (Figure 1).

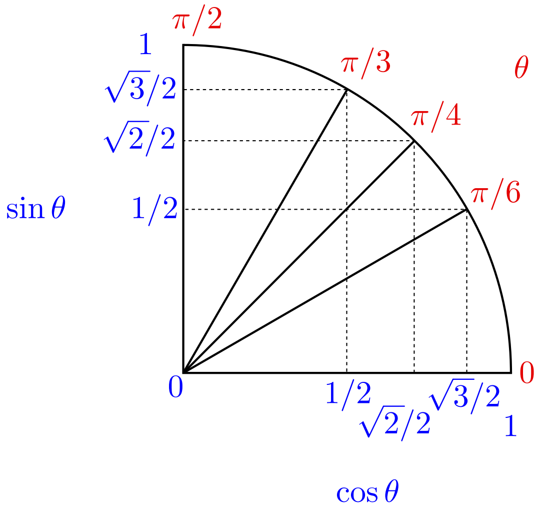

Figure 1: All you will ever need to know, in picture form!

Why am I making such a big deal out of this? Well, as the captions hint, this is pretty much all you will ever need to memorize when it comes to sine and cosine, beyond the basic definition. Technically, if you know the definition of sine and cosine, you don't need to memorize this table; after all, you figured out this table from the basic definition in last week's assignment. In practice, however, these special angles come up so often that it's very helpful to know these cold. You don't want to spend fifteen minutes drawing pictures of triangles and working through the Pythagorean theorem every time you need to evaluate $\sin(\pi/3)$.[1]

Problem 2. Your mission, should you choose to accept it,[2] is to take as long as you need to memorize one of the above tables and/or picture (whichever

[1] Trust me on this.

[2] The foregoing learning-enhancement task (the "ASSIGNMENT") shall remain the sole responsibility and charge (the "MISSION") of any learners who are party to said ASSIGNMENT (the "STUDENTS"), regardless of otherwise extenuating circumstances, such as, but not limited to, non-acceptance of the MISSION by the STUDENTS; prandial activities of any canine, feline, bovine, ursine, or other non-*homo sapiens* companion organisms, even when said prandial activities are conducted in reference to the ASSIGNMENT; alignment or non-alignment of heavenly bodies; or destructive behavior by egregiously sized, genetically anomolous amphibious organisms. The author of the ASSIGNMENT (the "TEACHER") shall under no circumstances assume any liability for any adverse affects which may or may not arise as a result of completing the MISSION, including, but not limited to, hypertension, sleep apnea, halitosis, somnambulance, rheumatism, gnosticism, or anything else ending in -ism.

is easiest for you to remember), and then evaluate each of the expressions found on the last page of the assignment, *without peeking*.

The reason this is all you need to know has to do with a *special amazing fact*...

2 Reference angles

reference angles

Every angle θ has a corresponding *reference angle* θ_{ref}, which is the smallest positive angle between θ's terminal ray and the x-axis. Figure 2 shows some examples.

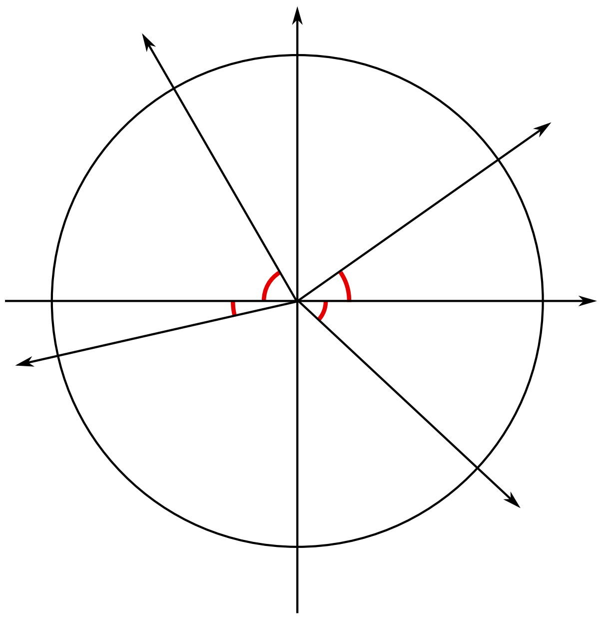

Figure 2: Reference angles

Problem 3. Let's figure out how to compute reference angles. Note, drawing pictures can help a lot!

(a) Suppose $\theta \in [0, \pi/2]$. What is θ_{ref}?

(b) Suppose $\theta \in (\pi/2, \pi]$. How can you determine θ_{ref}? For example, what is the reference angle for $\theta = 2\pi/3$?

(c) Now suppose $\theta \in (\pi, 3\pi/2]$. What is θ_{ref}?

(d) What is θ_{ref} when $\theta \in (3\pi/2, 2\pi)$?

(e) How would you determine the reference angle for an angle bigger than 2π? For example, what is θ_{ref} for $\theta = 98\pi/3$?

Problem 4. Find θ_{ref} for each of the following angles.

(a) $9000°$

(b) $11\pi/3$

(c) $-17\pi/9$

(d) $-\pi/20$

special amazing fact!

And now, for the *special amazing fact*:

$$\sin\theta = \pm\sin(\theta_{ref})$$
$$\cos\theta = \pm\cos(\theta_{ref})$$

That is, in order to find the sine or cosine of an angle θ, all you have to do is find the reference angle θ_{ref}, find the sine or cosine of that, and then decide whether the answer should be positive or negative! This is why we only ever need to be able to compute the sine or cosine of angles between 0 and $\pi/2$.

Problem 5. In your own words, explain why the *special amazing fact* is true. (It is pretty special, but it actually isn't all that amazing if you think about the definitions of sine, cosine, and reference angles.)

determining signs

Now, the one remaining important question: how to decide whether the answer should be positive or negative? Well, that's not too hard.

Problem 6. Complete Table 3, which shows where cosine and sine have positive and negative results. Recall that the quadrants are labelled I–IV counterclockwise starting from the upper right. (Remember the fundamental fundamentals. . .)

Problem 7. Compute by finding the reference angle. Don't use a calculator.

(a) $\cos(90000\pi)$

In quadrant...	$\sin\theta$ is...	$\cos\theta$ is...
I	+	+
II		
III		
IV		

Table 3: Signs of sine and cosine

(b) $\sin(77\pi/6)$

(c) $\cos(-983\pi/4)$

(d) $\sin(93\pi/2)$

Problem 2, continued. Evaluate each of the following, *without peeking.* You can peek at last week's assignment to remind yourself of the definitions of the other trigonometric functions if you need to.

(a) $\sin(\pi/3)$

(b) $\cos(\pi/2)$

(c) $\cos(\pi/6)$

(d) $\sin(\pi/6)$

(e) $\cos 0$

(f) $\sin 0$

(g) $\cos(\pi/4)$

(h) $\tan(\pi/2)$

(i) $\sec(\pi/3)$

(j) $\csc(\pi/6)$

