#11: Inverse trigonometric relations

November 14, 2008

a shady character

Problem 1. A shady-looking character slinks up next to you and whispers, "Pssst. Hey kid. I'll tell you a real number x, and I'll bet you \$50 you can't tell me an angle whose sine is x." Should you take the bet?

Problem 2. What if the shady character promises to tell you a real number between 0 and 1?

Problem 3. Now the shady character wants to bet you \$500 that if he gives you a real number x between -1 and 1, you can't tell him *fifty different* angles whose cosine is x. Should you take the bet this time?

1 Inverse trig relations

reversing trig operations

Now that we know how to find the sine or cosine of an angle (or tangent, secant, cosecant, or cotangent), we often want to do the reverse: that is, given some number x, find an angle whose sine or cosine is x. In other words, we want to find *inverse functions* for sine and cosine. However, there are a few problems to think about first.

Problem 4.

(a) Clearly, the domain of the sine function is $\mathbb{R}$, the set of all real numbers, since we know how to find the sine of *any* angle. Suppose we specify that the codomain of sine is also $\mathbb{R}$. In this case, is sine surjective (onto)? Why or why not?

(b) Now suppose we specify the codomain of sine to be the interval $[-1, 1]$. Now is sine surjective?

(c) Is the sine function injective (one-to-one)?

(d) What can you conclude about the existence of an inverse sine function?

inverse relations, not functions

Because of this, instead of inverse *functions*, we talk about inverse *relations* for sine and cosine; for a given real number $x \in [-1, 1]$, these inverse relations produce a *set* of angles (an inverse *function* would only be allowed to produce a single angle), all of whose sine (or cosine, or tangent, *etc.*) is x.

2 The inverse cosine relation

arccos

The inverse cosine relation is usually called arccosine and abbreviated arccos. (When writing arccos in an equation, be sure to use the LaTeX command `\arccos` so that it is typeset upright, like this: $\arccos(0.5)$ instead of like this: $arccos(0.5)$. The same goes for arcsin and arctan.) For any input real number $x \in [-1, 1]$, it outputs the set of all angles θ for which $\cos(\theta) = x$.

To figure out how arccosine should work, let's take as an example $x = 1/2$.

Problem 5. Find an angle θ for which $cos(\theta) = 1/2$. That is, find *one of* the angles output by $\arccos(1/2)$.

Problem 6. Given the angle θ you found in the previous problem, what's one quick way to find a different angle which also has a cosine of $1/2$? (Hint: see problem 8a from Assignment 9!)

Problem 7. Given the same θ again, what's *another* quick way to find a different angle with a cosine of $1/2$? (Hint: see problems 6 and 7 from Assignment 9!)

computing arccos

It turns out that this is enough. In order to compute $\arccos(x)$, you only need to somehow find a single angle θ for which $\cos(\theta) = x$; all the rest can be found by applying the above two tricks. Putting it all together:

$$\arccos x = \pm\theta + k \cdot 2\pi \tag{1}$$

where $\cos(\theta) = x$ and $k \in \mathbb{Z}$.

That is, given one angle whose cosine is x, we can (optionally) take the negative of it, and add or subtract 2π as many times as we want; all the angles we get in this way will have a cosine of x.

Problem 8. List six different angles in $\arccos(\sqrt{2}/2)$.

Problem 9. Find all the angles between 2π and 3π which have a cosine of $-1/2$.

3 The inverse sine relation

arcsin

The inverse sine relation, as you might have guessed, is called arcsine (abbreviated arcsin).

We won't go through the details, but a process very similar to the one for arccosine can be used to find angles which all have the same sine. The only difference is that instead of taking the negative of θ (an angle and its negative never have the same sine), we can subtract θ from π to get another angle with the same sine. That is:

$$\arcsin x = \theta + 2k\pi \text{ OR } (\pi - \theta) + 2k\pi \tag{2}$$

where $\sin(\theta) = x$ and $k \in \mathbb{Z}$.

Problem 10. Find all the angles between 0 and 4π which have a sine of 1/2.

4 Inverse trig relations and your calculator

calculators don't like printing infinite things

So, where are arccos and arcsin on your calculator? Look hard as you may, you will not find them. This is for a very practical reason: having a button which makes your calculator print out an infinite number of answers would be, shall we say. . . Not Very Useful.

inverse trig functions

Instead, your calculator has buttons labeled sin^{-1}, cos^{-1}, and tan^{-1}, which are similar but slightly different. Let's figure out what they do.

Problem 11. Try typing $\mathsf{sin}^{-1}(1/2)$ into your calculator. What answer do you get? What is this in terms of π (hint: try dividing by π)?

Problem 12. Use your calculator to compute the sin^{-1} of a bunch of other values. All the answers should fall within a certain range. What do you think the range of possible answers is?

In other words, sin^{-1} is actually an inverse *function* for sine on a very restricted domain, and similarly for the others.

But this is very useful: given x, it lets you compute *one* angle whose sine is x. This is all we really need. As we know from equation (2), once we know one such angle θ, figuring out all the others is simple.

5 Beware!

beware!

Beware!

sin^{-1} (and likewise cos^{-1} and tan^{-1}) are quite possibly the *most horribly chosen mathematical notation* in the *history of the world*.[1] Let me explain why.

- Normally, raising something to the power of negative one means to take the reciprocal. For example, $5^{-1} = 1/5$.
- So, you *might think* that $\sin^{-1} x$ means $1/(\sin x)$, but if you thought that you would be HORRIBLY WRONG. $\sin^{-1}$ is *just a notation* for the inverse sine function; it has *nothing to do with raising anything to the negative first power*!. It might as well be $\sin^{\oplus} x$ or $\sin^{\rightarrow\perp\#} x$.
- In particular, $\sin^{-1} x$ and $1/(\sin x) = \csc x$ are *not the same thing*.
- To make things even worse, as you will learn next week, $\sin^2 x$ *is* actually used as a notation for the square of $\sin x$. In other words:

  $$\sin^2 x = (\sin x)^2 \tag{3}$$

  but

  $$\sin^{-1} x \neq (\sin x)^{-1} \tag{4}$$

- I would like to find whoever made up this notation for inverse trig functions and make them pay $10 to every student who has ever been confused by it. The only problem is they probably don't have A GAZILLION DOLLARS.

6 Inverse trig relations and problem-solving

solving equations with arctrig relations

We can use arccos and arcsin to solve problems involving sine and cosine. If you have an equation of the form

$$\sin\theta = x,$$

[1] like, seriously.

you can solve for θ by taking the arcsin:

$$\theta = \arcsin x.$$

Of course, the same goes for cosine. You can then use your calculator to compute $\sin^{-1} x$ or $\cos^{-1} x$, and then use equations (2) and (1) to find other solutions as necessary.

Problem 13. Find all angles $\theta \in [0, 2\pi)$ for which

$$3(\cos(\theta))^2 + 2 = 4.$$

Problem 14. The triangle shown in Figure 1 is a right triangle with legs of length 3 and 4. What is θ? (Give your answer in radians, rounded to three decimal places.)

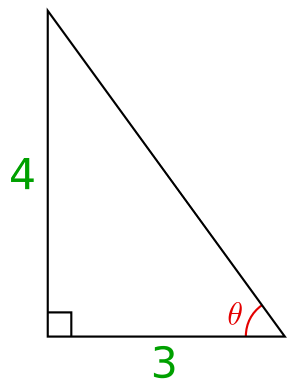

Figure 1: A triangle.

