#15: Vectors

January 17, 2009

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rectangular/polar duality

Last week, you studied the duality between *rectangular* and *polar* coordinates; as you learned, they are two different ways of representing the same information (position in two-dimensional space), and we can easily convert back and forth between the two representations.

But why would we care that there are two different representations? Can't we just pick one and use that? The answer, of course, is that each representation is useful for different things, and is related to other mathematical concepts in different ways.

This rectangular/polar duality comes up in two other particular places we're going to study. The first, vectors, are a way of formalizing the ideas of motion and force. The second may be unexpected: complex numbers also exhibit this rectangular/polar duality! If you've seen complex numbers before, you probably learned only about the rectangular representation: $a + bi$. But complex numbers can also be thought of in polar terms, which we will study in an upcoming assignment.

1 The Racecar Game

vroom!

Before starting this assignment (or at least at some point during the week), try playing the racecar game, which is described in a second document linked from the webpage!

2 Vectors as Arrows

vector definition

A *vector* is a mathematical object that represents an *amount* (usually called the *magnitude*) as well as a *direction*. For example, each segment that a car traveled in the racecar game was a vector: it had an amount (how far the car traveled) and a direction. A force can also be represented as a vector: it has an amount (the strength of the force) as well as a direction (the direction in which the force is pushing).

Graphically, we can represent vectors using arrows. The *length* of the arrow represents the magnitude of the vector, and the direction of the arrow represents the direction of the vector. The end with the arrowhead is called the *head*, and the other end is called the *tail*.

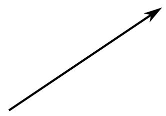

Figure 1: A vector.

Since vectors just have a direction and a magnitude, it doesn't matter where a vector starts or ends. Moving a vector around doesn't change it at all. Figure 2 shows three vectors which are all equal.

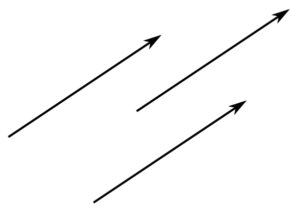

Figure 2: Three equal vectors.

3 Vector Addition

We can add two vectors together to produce another vector, and this means exactly what you might think. If you travel 40 feet in a certain direction, and then 50 feet in another direction, what how far and in what direction did you travel altogether? If two people are pushing on a box in different directions and with different strengths, in what direction will the box move?

vector addition graphically

Graphically, vector addition can be represented by putting the two vectors to be added head-to-tail; then their sum is the vector from the head of the first to the tail of the second, as shown in Figure 3. We put a little arrow over a variable (like $\vec{a}$) to indicate that we are talking about a vector. To make something like $\vec{a}$ in LaTeX, you can type `\vec{a}`.

We can use the Law of Cosines and Law of Sines to compute the sum of two vectors.

Problem 1. Consider Figure 4, which shows two vectors, one with length 12 and angle 50°, and the other with length 5 and angle −30° (as usual,

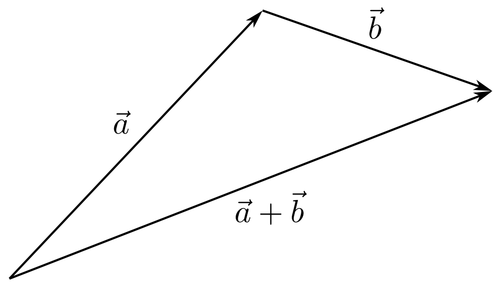

Figure 3: Adding the vectors $\vec{a}$ and $\vec{b}$.

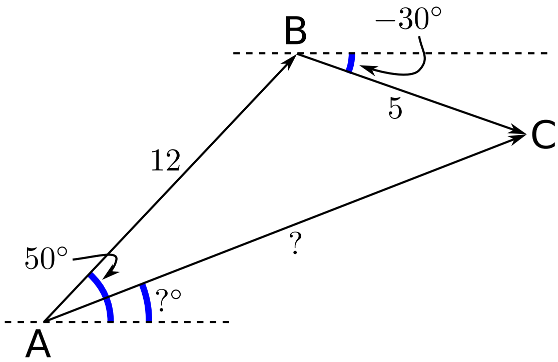

Figure 4: Adding two vectors using the Law of Cosines

we measure vector direction as angles from the positive x-direction). If we add these two vectors, we get the third vector shown on the bottom. Let's compute the magnitude and direction of this vector.

(a) First, find angle B.

(b) Now use the Law of Cosines to find the length of AC.

(c) Use the Law of Sines to find angle BAC.

(d) Finally, use this to find the direction of vector AC (marked ?° in the diagram).

4 Vectors and rectangular coordinates

Adding vectors using the Law of Cosines in this way is rather cumbersome. Isn't there a better way? As it turns out, there is!

A vector has a *magnitude* and a *direction*. . . a *length* and an *angle*. . . doesn't this sound familiar? It's just like polar coordinates! In fact, it *is* polar coordinates. So maybe there's a way to represent vectors using rectangular coordinates, too?

vector decomposition

Sure enough, there is! Any vector $\vec{v}$ can be decomposed into two vectors, one vertical and one horizontal, whose vector sum is $\vec{v}$. An example is shown in Figure 5.

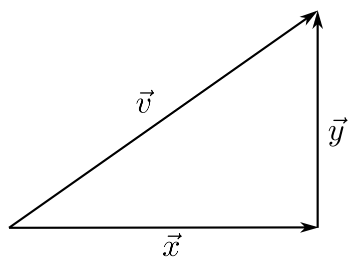

Figure 5: Decomposing $\vec{v}$ as $\vec{x} + \vec{y}$

But you already know how to do this from last week's assignment! It's just converting from polar to rectangular coordinates.

Problem 2. Convert vector AB in Figure 4 to rectangular coordinates.

Problem 3. Convert vector BC in Figure 4 to rectangular coordinates.

The nice thing is that now the vectors are very easy to add: just add the x-components, and add the y-components.

Problem 4. Add the vectors from the two previous problems using rectangular coordinates.

Now we can convert back to polar coordinates: you already know how to do this, too.

Problem 5. Convert the vector from Problem 4 back into polar coordinates. Does it agree with your answers to Problem 1?

a Hard(ing) problem

Problem 6. Adler, Bartholomew, and Custer are pushing a large marble statue of President Harding on wheels. Adler is pushing at an angle of 20° with a force of 50 Newtons. Bartholomew is pushing at an angle of 25° with a force of 40 Newtons. Custer doesn't want them to move the statue, so he is pushing back at an angle of 185° with a force of 60 Newtons. In what direction and with what force does the statue move? Show your work, and express your answers rounded to the nearest three decimal places.

(Hint: to find the total force you can just add the vectors corresponding to each person's pushing. You can either use the Law of Cosines (twice!) or (recommended) convert to rectangular coordinates first, add componentwise, and convert back to polar coordinates.)

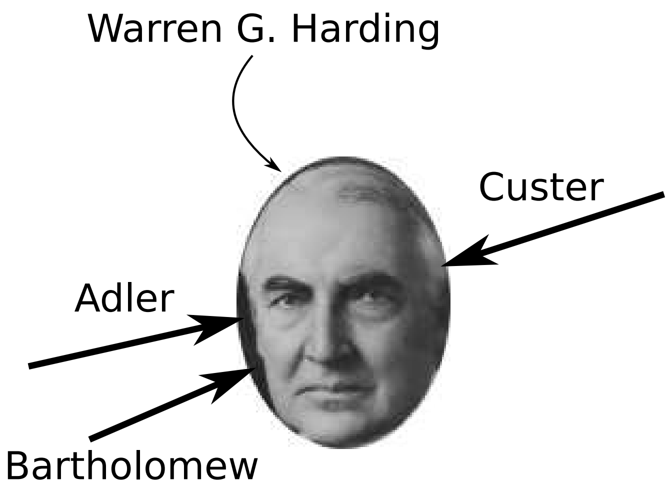

Figure 6: A statue-pushing match

