#18: Matrices

February 7, 2009

The *complex numbers* you learned about in previous weeks are a particular generalization of the concept of "number;" so, too, are *matrices*. You can think of a matrix as a sort of "multi-dimensional number."

Now, why might this be useful? Matrices, as it turns out, have a ton of applications; we'll look at a few next week. For example, matrices can be used to represent certain sorts of transformations in two or three dimensions, so they are used extensively in computer graphics. (In fact, it turns out that the graphics card inside your computer is essentially just a glorified piece of hardware for doing lots of matrix multiplications, very quickly, in parallel!) Matrices can also be used to represent systems of linear equations, or sets of transition probabilities in multi-state systems, or pretty much anything having to do with sets of data having multiple dimensions.

Before we get there, however, we'll spend this week learning some matrix basics.

Be sure to look at the LaTeX notes in the last section of this assignment for some tips on typing up your solutions!

1 The basics

$\boxed{matrix}$

> A *matrix* is a rectangular array of numbers. We say that a matrix M is an $m \times n$ matrix when it has m rows and n columns, and call m and n the *dimensions* of M.

For example, this is a 2×3 matrix:

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & -9 \end{pmatrix} \tag{1}$$

A matrix is *square* if it has the same number of rows and columns.

row, column   Repeat this phrase to yourself: "row, column." We always refer to the *rows* of a matrix first, and the *columns* second. (This is completely arbitrary, of

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course. It *could* have been the other way around.[1]) This applies to matrix dimensions (the dimensions $m \times n$ mean *m rows* and *n columns*) and also, as you will see, to remembering how matrix multiplication works.

subscript notation We can use subscript notation to refer to particular entries in a matrix: the notation M_{ij} refers to the entry of matrix M in row i and column j.

Problem 1. Let A be the matrix in equation (1). What is A_{13}?

$\boxed{\textit{transpose}}$

> The *transpose* of a matrix M, denoted M^T, is the matrix M with the rows and columns switched. That is, $(M^T)_{ij} = M_{ji}$.

You can think of the transpose as flipping the matrix along a diagonal line going from the top left to bottom right. For example,

$$A^T = \begin{pmatrix} 2 & 4 \\ 3 & 1 \\ 5 & -9 \end{pmatrix}.$$

2 Matrix arithmetic

matrices as generalized numbers That's right, we can do arithmetic with matrices—in other words, we can treat them as a particular kind of "generalized number." Why might we want to do this? There are many good reasons, some of which you'll see later on in the assignment. For now, let's see how matrix arithmetic works.

2.1 Matrix addition

easy peasy Matrix addition is very simple, and works in exactly the way you might guess. We can only add two matrices with the same dimensions; but assuming we do have two matrices A and B with the same dimensions, we can add them simply by adding corresponding entries. For example,

$$\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 2+5 & 3+0 \\ 4-6 & -1+3 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -2 & 2 \end{pmatrix}$$

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[1]But it isn't.

Problem 2. Suppose we have the following matrices:

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & -9 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 6 \\ 2 & 1 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \end{pmatrix}$$

Evaluate each of the following.

(a) $A + C$

(b) $B + C$

(c) $B + A^T$

(d) $A + B^T + C$

Problem 3. For two matrices X and Y, is $X + Y$ always the same as $Y + X$? Explain why, or give a counterexample.

2.2 Matrix multiplication

Multiplying two matrices, on the other hand, is *not* quite as straightforward as addition!

say what? To multiply two matrices, the number of *columns* of the first matrix must be the same as the number of *rows* of the second matrix. Let's say that we have two matrices, X, which is $m \times k$, and Y, which is $k \times n$. Then their product, denoted XY, will be an $m \times n$ matrix. Here is how to determine the elements of the matrix product XY: to get $(XY)_{ij}$ (the entry in the ith row and jth column), take the ith row of X and the jth column of Y, multiply their corresponding elements, and add the results.

Now, maybe you are confused by that description; I wouldn't blame you. But it's really not so bad once you get the hang of it; let's go through an example.

matrix multiplication example Suppose we have the matrices

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -2 & 1 & 9 \\ 5 & 4 & 2 & 6 \\ 1 & -3 & 0 & -1 \end{pmatrix}.$$

X has dimensions 2×3, and Y has dimensions 3×4. Since the number of columns of X (3) equals the number of rows of Y (also 3), we can multiply

them, and the result will be a 2×4 matrix. Think to yourself: "2×3 and 3×4—the 3's match up, so they disappear, leaving 2×4."

Now, to determine the entry in the first row and first column of the product XY, we look at the first row of X and the first column of Y, here shown highlighted in blue and red:

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -2 & 1 & 9 \\ 5 & 4 & 2 & 6 \\ 1 & -3 & 0 & -1 \end{pmatrix}.$$

We now take these two lists of three numbers, multiply them element-by-element, and add the results:

$$1 \cdot 0 + 2 \cdot 5 + 3 \cdot 1 = 0 + 10 + 3 = 13.$$

So far, we know that the matrix product XY looks like this:

$$XY = \begin{pmatrix} 13 & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

Now let's compute $(XY)_{12}$ (highlighted in red above). Since we are trying to compute the entry of XY in the *first row* and *second column*, we take the *first row* of X, and the *second column* of Y:

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -2 & 1 & 9 \\ 5 & 4 & 2 & 6 \\ 1 & -3 & 0 & -1 \end{pmatrix}.$$

Multiplying them pairwise and then adding yields

$$1 \cdot -2 + 2 \cdot 4 + 3 \cdot -3 = -2 + 8 + -9 = -3.$$

Now XY looks like

$$XY = \begin{pmatrix} 13 & -3 & ? & ? \\ ? & ? & ? & ? \end{pmatrix}.$$

Getting the hang of it?

Problem 4. Go back and read the paragraph at the beginning of section 2.2. Does it make sense now?

Problem 5. Finish computing the matrix product XY. (You don't have to show all the work, just the final answer.)

Problem 6. Use the matrices A, B, and C defined in Problem 2, and also

$$D = \begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}.$$

Compute each of the following:

(a) AB

(b) AC

(c) BD

(d) DE

(e) ED

Problem 7. Suppose X and Y are square matrices. Is XY always the same as YX? Explain why, or give a counterexample.

matrix multiplication is not commutative

What you found in Problem 7—that matrix multiplication is not *commutative*—is one of the biggest ways that matrices are different from any other sort of "number." With real numbers, you are used to being able to switch around the order of things being multiplied, but you have to be very careful: you cannot do this with matrices!

matrix multiplication is associative

It turns out, however, that matrix multiplication is *associative*: for any matrices X, Y, and Z, as long as they have dimensions that match up properly, it is always true that $(XY)Z = X(YZ)$. That is, when doing more than one matrix multiplication, it doesn't matter which multiplication we do first, as long as we keep them in the right order. This means that we can write things like $ABCDE$ instead of $((AB)C)(DE)$ or $A(B(C(DE)))$ or $((AB)(CD))E$ since they are all the same.

Problem 8. Can you find a 2×2 matrix I, which when multiplied by any other 2×2 matrix X, yields X? That is, $IX = X$ for any 2×2 matrix X. I is called the 2×2 *identity matrix* (sometimes also written I_2).

Problem 9. What is the 3×3 identity matrix, I_3? In general, what does the $n \times n$ identity matrix I_n look like?

Problem 10. There is no such thing as a 2×3 identity matrix. Why not?

3 Inverses and the determinant

Problem 11. Multiply the following two matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$

What do you get? Why is this interesting?

inverse

> The *inverse* of a square matrix A, written A^{-1}, is a matrix which when multiplied by A results in the identity matrix: $AA^{-1} = A^{-1}A = I$.

Problem 12. As it turns out, not all matrices have an inverse. But this should not be too surprising—why not? (Hint: think about the real numbers.)

solving matrix equations

Matrices which have an inverse are called *invertible*, and matrices which do *not* have an inverse are called *singular*. Why do we care whether a matrix is invertible? Well, remember what you do in algebra to solve an equation like $3x = 12$: you multiply both sides by $1/3$, the *inverse* of 3. In the same way, inverting matrices allows us to solve *matrix equations* like $AX = Y$ (where A, X, and Y are all matrices)—if A is invertible, we can multiply both sides of the equation by A^{-1} to get $X = A^{-1}Y$.

So, we would like a way to be able to tell whether a matrix has an inverse, and, if it does, to be able to compute it. Interestingly, both of these require something called the *determinant*.

determinant

> The *determinant* of a matrix M, denoted $\det M$, is a number computed from M in a particular way (to be described later), which has the special property that M is invertible *if and only if* $\det M \neq 0$.

That is, to check whether a matrix M has an inverse, just compute its determinant—if the determinant is zero, M is singular (that is, not invertible); it the determinant is anything other than zero, M is invertible. Nifty, huh? Oh, except I haven't told you how to compute the determinant yet!

This week, I'll only show you how to compute the determinant of a 2×2 matrix.[2] Suppose we have the matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the determinant of X (which we can abbreviate using vertical bars instead of parentheses around the matrix elements) can be computed by

$$\det X = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Problem 13. What is the determinant of matrix A from Problem 11?

Problem 14. Write down a 2×2 matrix with determinant 5.

Now let's see how to compute the inverse of a 2×2 matrix. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Problem 15. For each of the following matrices, compute its inverse, or state that it has none.

(a) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (*Hint: remember the Pythagorean Identity!*)

Problem 16. Solve this matrix equation for X:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X = \begin{pmatrix} 7 \\ -2 \end{pmatrix}.$$

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[2]If you're extra good, next week I might show you how to compute the determinants of larger ones...

4 LaTeX notes

- To typeset the size of a matrix (for example, "$m \times n$"), you should write something like `$m \times n$`; do not just write an 'x', like `$m x n$`. Do these look the same to you: $m \times n$, mxn? I didn't think so.

- You can typeset a matrix using the `pmatrix` environment. Elements within a row are separated by `&`; rows are separated by `\\`. For example, you could typeset the following matrix:

$$\begin{pmatrix} 4 & 5 & x+2 \\ 9 & 0 & \pi \\ 6+5i & \sqrt{3} & -2 \end{pmatrix}$$

with this code:

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\[
  \begin{pmatrix}
    4 & 5 & x + 2 \\
    9 & 0 & \pi \\
    6 + 5i & \sqrt{3} & -2
  \end{pmatrix}
\]
```

To get a matrix with vertical bars instead of parentheses, you can use `vmatrix` instead of `pmatrix`. (There is also `bmatrix` to make a matrix with square brackets, but you won't need that.)

- When using matrix subscript notation, remember to use curly braces around the subscripts; otherwise, only the first digit will be included in the subscript. For example, `M_12` is wrong, since it produces this output: $M_1 2$. Correct would be `M_{12}`, which produces M_{12}.

- When typesetting the determinant operator, you should use the special LaTeX command `\det` instead of just writing `det` (just like you use `\cos` instead of `cos`). See the difference: $\det M$ (correct), $detM$ (incorrect).

- If anything is still not clear, take a look at the LaTeX source for this assignment!