#24: Series

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1 Sigma notation

series

A *series* is the sum of the terms in some sequence. For example, if we have the simple arithmetic sequence 1, 2, 3, 4, 5, ..., the corresponding series is $1 + 2 + 3 + 4 + 5 + \dots$ and so on. Of course, adding up this particular series is meaningless, since it goes on forever. But it is often useful to talk about *partial sums* of a series, where we add up some of the terms and then stop. In general, we will use the notation S_n to denote the sum of the first n terms of a series. For example, for the series just mentioned, S_4 would be equal to $1 + 2 + 3 + 4 = 10$.

Problem 1. Consider the arithmetic sequence with $a = 5$ and $d = 2$. What is S_6?

Σ notation

Sigma notation can be used to concisely represent a series. Instead of explaining how it works, first I'll just give some examples:

$$\sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = 10$$

$$\sum_{k=1}^{4} 2k = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 = 2 + 4 + 6 + 8 = 20$$

$$\sum_{j=5}^{7} j^2 = 5^2 + 6^2 + 7^2 = 25 + 36 + 49 = 110$$

$$\begin{aligned}\sum_{n=-5}^{-2} \frac{2n+1}{n^2} &= \frac{2 \cdot (-5) + 1}{(-5)^2} + \frac{2 \cdot (-4) + 1}{(-4)^2} + \frac{2 \cdot (-3) + 1}{(-3)^2} + \frac{2 \cdot (-2) + 1}{(-2)^2} \\ &= -\frac{9}{25} - \frac{7}{16} - \frac{5}{9} - \frac{3}{4} = -\frac{7571}{3600}\end{aligned}$$

Got that? The big thing that looks kind of like an E is not an E at all; it's a capital sigma, the 18th letter of the Greek alphabet.[1] The variable

[1] Sigma corresponds to our letter S, which stands for *sum*.

at the bottom of the sigma is the *index of summation*, and the numbers at the bottom and top are the *limits of summation*. $\sum_{k=a}^{b} f(k)$ can be read as "the sum, as k goes from a to b, of $f(k)$." The idea is that you take each value of k starting from a and counting *by ones* up to (and including) b, and substitute it into the right-hand formula; then you add up everything you got by doing the substitutions.

You can create sigma notation in LaTeX using the `\sum` command, followed by a subscript and a superscript for the parts below and above the Σ. For example, `\sum_{k=1}^{20} (k^2 + 3)` produces

$$\sum_{k=1}^{20}(k^2+3).$$

Problem 2. Now you try a few.

(a) $\displaystyle\sum_{k=1}^{5}(2k-1)$

(b) $\displaystyle\sum_{j=1}^{6} j^2$

(c) $\displaystyle\sum_{r=2}^{2} r\pi$

(d) $\displaystyle\sum_{g=5}^{2}(g+6)$

(e) $\displaystyle\sum_{j=1}^{4}\left(\sum_{k=2}^{3} jk\right)$

Series are extremely common, so it's important to be able to evaluate, use, and manipulate them comfortably. Sigma notation allows us to be precise about series, and to be able to see the "big picture" without getting lost in scads of ellipses. Writing

$$\sum_{k=1}^{10} k$$

is not only more concise than writing something like

$$1 + 2 + 3 + \cdots + 10,$$

but it can also be much easier to read and work with, once you get used to it.[2]

2 Some special series

Problem 3. What is

$$\sum_{k=1}^{n} 1 \quad ?$$

arithmetic series

Problem 4. This problem will guide you through finding a *closed form* (by which, in this context, I mean an equivalent expression that does not involve a sum) for the series

$$\sum_{k=1}^{n} k,$$

that is, the sum of the first n whole numbers. As you may remember from a few assignments ago, these partial sums (1, $1+2$, $1+2+3$, ...) are called the *triangular numbers*.

Let $S = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$; we want to find a nice expression for S in terms of n. Consider adding S to itself, but pair 1 with n, 2 with $(n-1)$, and so on; each pair of numbers will add up to $(n+1)$. Like this:

$$\begin{array}{rcccccccccc} S & = & 1 & + & 2 & + & 3 & + \cdots + & n \\ + \;\; S & = & n & + & (n-1) & + & (n-2) & + \cdots + & 1 \\ \hline 2S & = & (n+1) & + & (n+1) & + & (n+1) & + \cdots + & (n+1) \end{array}$$

(a) Use this last equation to solve for S. (*Hint:* how many copies of $(n+1)$ are there?

(b) Use your solution for S to find the 327th triangular number.

geometric series

Problem 5. This problem will guide you to finding a closed form for the sum

$$\sum_{k=0}^{n} r^k.$$

Let $S = r^0 + r^1 + r^2 + \cdots + r^n$.

[2] It's also much easier to impress your friends with!

(a) What is $r \cdot S$?

(b) What is $S - r \cdot S$? Simplify your answer as much as possible (hint: a lot of things should cancel each other).

(c) Note that you can factor $S - r \cdot S$ as $S(1 - r)$. From the previous part you should now have an equation that looks like $S(1 - r) = \dots$ where the dots are your answer to part (b). Now divide both sides by $(1 - r)$ to obtain a closed form for S.

Problem 6. What is
$$\sum_{i=0}^{8} 2^i?$$

Problem 7. Simplify: $2x + 4x^2 + 8x^3 + 16x^4 + 32x^6$.

3 Properties of sigma notation

It's also important to investigate a few general properties of sigma notation, and see how we can use these properties to manipulate series. Suppose that $f(k)$ and $g(k)$ are any functions of k, and c is any constant (that is, an expression that does not involve k—in particular, it doesn't have to be just a number; it could involve *other* variables so long as it does not involve k). Then the following properties always hold:

$$\sum_{k=a}^{b} c = (b - a + 1) \cdot c \qquad \text{(sum of constants)} \qquad (1)$$
$$\sum_{k} c \cdot f(k) = c \sum_{k} f(k) \qquad \text{(distributivity)}^3 \qquad (2)$$
$$\sum_{k} [f(k) \pm g(k)] = \sum_{k} f(k) \pm \sum_{k} g(k) \qquad \text{(associativity)} \qquad (3)$$

These might look scary, but they're actually quite simple. The first property (equation (1)) expresses what happens when we add up a constant

[3] By the way, the fact that there are no limits of summation in this equation and the next isn't a typo; it just means that these equations hold independently of any particular limits of summation.

expression: since it doesn't involve k, it doesn't change, and we just add up $(b-a+1)$ copies of it (you should convince yourself that there are $(b-a+1)$ numbers from a to b inclusive). For example,

$$\sum_{k=2}^{5} 4e = 4e + 4e + 4e + 4e = 16e.$$

Problem 8. What is

$$\sum_{j=9}^{20} 6?$$

The second property (equation (2)) says that we can factor any constant expression out of a sum: this is just the normal distributive property of multiplication over addition. For example, saying that $(2+4+6+8+10) = 2 \cdot (1+2+3+4+5)$ is the same as saying that

$$\sum_{k=1}^{5} 2k = 2\sum_{k=1}^{5} k.$$

The last property (equation (3)) says that we can break up a sum of sums (or differences) into the sum (or difference) of separate sums. Really, all this says is that we can change the order of addition without affecting the result. For example, $(1+5)+(2+6)+(3+7) = (1+2+3)+(5+6+7)$; equivalently, we can say that

$$\sum_{k=1}^{3}(k+(k+4)) = \sum_{k=1}^{3} k + \sum_{k=1}^{3}(k+4).$$

However, you should note that the corresponding property does *not* hold for multiplication:

$$\sum_{k} f(k) \cdot g(k) \neq \left(\sum_{k} f(k)\right) \cdot \left(\sum_{k} g(k)\right). \tag{4}$$

That is, a sum of products is *not* the same thing as a product of sums!

Problem 9. Why not? Give an example to illustrate equation (4).

These properties are fairly simple, but in conjunction with a few special series we know (Problems 3–5), we can already use them to tackle some more complex series. For example:

$$\begin{aligned}
\sum_{k=1}^{n}(2k+3) &= \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 3 && \text{property (3)} \\
&= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3 && \text{property (2)} \\
&= 2 \cdot \frac{n(n+1)}{2} + 3n && \text{Problem 4, property (1)} \\
&= n^2 + n + 3n = n^2 + 4n. && \text{simplify}
\end{aligned}$$

Problem 10. Let Q represent the arithmetic sequence with $a = 3$ and $d = 2$. Find a closed-form expression for the sum of the first n terms of Q. (*Hint:* remember, the kth term of Q can be written as $a+(k-1)d = 1+2k$. Write down the appropriate sum and then use sigma notation properties to simplify, as in the example above.)

