The Math Less Traveled

(working title)

Chapter 0: Preface / Boolean Algebra
draft version 2
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This document includes both the preface and first (zeroth) chapter. Chapter 0 covers Boolean algebra and first-order logic—topics which provide a basis for the rest of the book and its emphasis on problem-solving and proof, but are nonetheless interesting topics in and of themselves. The present document is a revision of the original published draft, incorporating many suggestions from Stephen Gilberg, B. Tarkington, Tom White, and Rick Yorgey, as well as many of my own changes. It is still incomplete; in particular, I plan to add some problems dealing with applications of Boolean algebra to logic circuitry, and of course I am also happy to incorporate additional suggestions from test readers.

Preface

What is mathematics all about?

First, allow me to tell you some things it *isn't*: mathematics is not just a random collection of useful formulas. It's not about memorization, or computation. It's not about learning skills that will be useful to you someday if you happen to get one of the jobs listed down the side of those posters with the grid showing what areas of math are used in what sorts of jobs.[1] At root, it's not even about numbers! Mathematics is no more about any of these things than history is about memorizing dates, or art is about learning how to use a pencil.

So what *is* it about? Fundamentally, mathematics is about *pattern*—discovering, exploring, and describing the patterns built into the fabric of the universe. To be a mathematician—and by that I mean anyone who does mathematics—is to go on fantastic voyages in search of pattern, truth, and beauty, voyages filled with surprise, wonder, and yes, fun!

This book is an invitation to leave the shallow harbor of memorization and drudgery, and to venture out into the vast ocean of mathematical wonder. It's not a textbook, nor an introduction to any particular subject; it's more like (to continue the metaphor) a pleasure cruise with a number of island stops, each providing a glimpse of a particular mathematical topic. Some (such as sequences and series) are often studied in high school; others (such as group theory) are not; but all are beautiful in their own ways, and none require calculus to understand.

[1] You probably know the sort of poster I'm talking about. Don't get me started on those.

Notes to the reader

One of the things which I think makes this book unique is that it makes *you* do a lot of the work. But wait—don't run away! Allow me to explain. There are more than ??? problems of varying difficulty scattered throughout the text, in addition to more traditional end-of-the-chapter exercises. The problems are integral parts of the text and prompt you to play with, discover, or prove important mathematical results for yourself. Of course, no one can stop you from just skipping over the problems[2]—and I don't deny that you could probably still learn a few things that way. But the only way to *really* get the full benefit of this book is to try the problems as you encounter them, since the emphasis is on guiding you to discover things for yourself, rather than presenting them to you as facts. But don't panic! There is plenty of guidance along the way (including hints for selected problems, and an entire chapter dedicated to proofs and problem-solving methods), and your efforts will be rewarded by a far richer understanding of the material than you could gain just by reading.

At this point you probably have one of three attitudes, to which I have one of three corresponding responses:

(a) *This sounds awesome. Bring it on!*

Great! You're going to have a lot of fun.

(b) *This sounds hard. I don't know if I'll be able to do it.*

Nonsense! You can do it! Most worthwhile things are hard, but that's no reason not to do them.

(c) *This sounds like a lot of work. I hate work. I want all my knowledge served to me on a silver platter. In a silver cup. With a silver spoon. Oh, and if you could fold my laundry for me while you're at it, that would be great.*

Go away. This book is not for you.

[2]But Santa Claus knows.

Each problem comes with a difficulty rating:

- ($\star$) indicates a problem which should be fairly straightforward to solve.
- ($\star\star$) indicates a problem which may take more thought than a level 1 problem. A level 2 problem may range in difficulty from slightly to moderately difficult.
- ($\star\star\star$) indicates a more difficult problem which will generally require patience and determination to solve.
- ($\star\star\star\star$) indicates a 'challenge' problem which most readers will probably find very difficult.

You should keep in mind, however, that the difficulty ratings are there to give you a general idea of what to expect, not to scare you off! The ratings are naturally subjective, and what one person finds difficult may seem easy to another, and vice versa. I would suggest that you at least try all the problems marked ($\star$)–($\star\star\star$) when you encounter them; but I won't blame you if you decide to initially skip problems marked ($\star\star\star\star$) and come back to tackle them later.

Hints for selected problems can be found at the end of each chapter. If you're stuck on a particular problem, the hint can give you a gentle push in the right direction. Not all problems have hints; if there is a hint for a particular problem, it will be indicated by the symbol ★.

Complete solutions for all the problems can be found in a special section in the back of the book. Of course, it's up to you to decide how you will use the solutions. I know how tempting it can be to just look up the answer as soon as you get a little bit stuck—but you'll learn a lot more if you avoid the temptation! My recommendation is to never look at the solution the first time you work on a problem, unless you're pretty sure you've solved it and want to check. If you're having trouble with a problem, don't look at the answer—put it down and come back later. This is how 'real' mathematics works: there's no 'Back of the Universe' where mathematicians can go look up the answer when they can't solve a problem! Try some different approaches. Work on it together with some other people. If you *still* aren't having any luck, *then* take a look at the solution—and make sure you really understand it before moving on. That's my recommendation; ultimately, however, the goal is for you to have fun and learn some things, so you should feel free to use the solutions in whatever way works best for you!

You will also find footnotes[3] sprinkled liberally throughout the book. Sometimes they'll be informative, sometimes justificatory,[4] but mostly they'll just be entertaining[5] reflections of my non-linear thought processes. I hope you'll enjoy them. But if you can't stand the footnotes, then don't read them.[6]

Notes to parents and educators

In my experience, elementary and secondary math curricula today focus on *skills*—equipping students with knowledge supposedly of use in our increasingly technical society. In consequence, math comes across to students as a collection of formulas, processes, and techniques to use in solving problems; little or no attention is given to beauty, exploration, discovery, and proof. For some students, this state of affairs is just fine. But there are others who really do feel stuck in the "shallow harbor of memorization and drudgery." The real tragedy is that they don't realize it's a harbor rather than a pond.

Most students do not get a real taste of the joy and beauty of mathematics for its own sake until they get to college—if they are still interested enough to pursue it. For many, it is too late, having decided in junior high or high school that math is not for them. This book is my attempt to convince some students that math can be beautiful, fun, and exciting . . . and that maybe it *is* for them after all.

[3] Like this.

[4] That's not a real word, is it?

[5] Or stupid, depending on your point of view.

[6] Hah, I dare you to try!

I haven't written this book with any one particular setting or use in mind, but I imagine that it could be used successfully in a home-school setting, in an extra-curricular math club, as enrichment in a high school mathematics course, or simply as a gift to a mathematically inclined student. Although many of the topics are not usually covered in high school curricula, the problem-solving techniques, mathematical insights, and confidence that students will gain will undoubtedly transfer to greater success—and, hopefully, greater enjoyment—in a regular mathematics course.

Final comments

Intrigued and want to learn more? Have a comment or a burning question you'd like to have answered? Come visit my website:

`http://www.mathlesstraveled.com`

You'll find links to other math-related websites, a list of errata for this book,[7] a blog where I regularly post bite-sized bits of interesting math, and a forum where you can discuss various math topics with me and others.

If there's one final thing I'd like to say, it's this: have fun! If you have half as much fun reading this book as I had writing it,[8] that's good enough for me.

About this book

This book was typeset using LaTeX, a markup-based document processing system which is in turn based on Donald Knuth's TeX typesetting system. LaTeX can be used to typeset many different types of high-quality

[7] AS IF this book will have any errors. Please.

[8] The mature, responsible side of the author's personality (which actually did all the work while the fun-loving, carefree side which had originally come up with the idea of writing a book daydreamed about fractals . . . not that it is bitter or anything) wishes to inform you that writing a book actually consists primarily of long, tiresome drudgery requiring persistence and hard work. It is not recommended unless you are slightly insane.

documents, and is the *de facto* standard in mathematics and science publishing.[9] It is also 100% free.[10] For links to more information, or to try it out, visit http://ctan.org.

Donald Knuth's Concrete family of fonts was used for setting the text, along with Hermann Zapf's AMS Euler font for $\mu_\alpha T^\hbar \sum_m \forall\tau\, \vec{\imath}\, \mathbb{C}(5)$.

Final final comments

What, are you still reading this preface? Shoo! Get on with the real book already!

[9] LaTeX is to Word as a fighter jet is to a station wagon: it's completely different and a lot harder to learn how to use well, but once you do . . .

[10] Both as in speech and as in beer.

Chapter 0

Boolean Algebra

So, here we go! We're all ready to embark on our fantastical journey through Mathematics.[1] We begin our journey with a tour of Boolean[2] algebra. If you've ever taken any classes in computer programming, you've probably seen Boolean algebra in one form or another;[3] otherwise, you probably haven't, which is a shame but easily remedied. Boolean algebra has an elegant simplicity and symmetry that simply shouldn't be missed. Plus, it's kind of fun.

You may be surprised to find that this, the first chapter in a book of mathematics, doesn't contain all that many numbers! But mathematics isn't just about numbers; it's really about pattern and structure, and that's exactly what Boolean algebra is: the study of the recurring patterns and structures found within the realm of logical reasoning.

0.1 Say what-ean algebra?!

Boolean algebra is named for the British mathematician George Boole, who pioneered the subject in the mid-1800's. In a nutshell, Boolean algebra is

[1] Oops, hold on just a minute, I forgot the sunscreen . . . okay, I'm back. Let's go.
[2] Just FYI, that's three syllables, not two.
[3] Even if you didn't know it at the time.

to logic as 'normal' algebra is to arithmetic—that is, Boolean algebra lets us reason abstractly about logic in the same way that the more familiar algebra you learned in school lets you reason abstractly about numbers.

In normal algebra, a *variable* such as x can have any one of an infinite number of possible values—such as 17, or -4.9, or $\pi/6$, or $\tan^{-1}(\sqrt{3+e})$, or who knows what else. In Boolean algebra, on the other hand, a variable can have one of *only two* possible values: *true* or *false* (which we often abbreviate T and F, respectively). That's it! No *maybes*, no *buts*, no $-F$ or $\sqrt{T+F^2}$ or F.TTFFTFT. Just T. Or F.

Problem 0.1 ($\star$). Suppose we have three Boolean variables: A, B, and C (we often use uppercase letters for Boolean variables, although there's nothing to stop us from using lowercase letters too[4]). How many different ways are there, in total, to assign a combination of values to A, B, and C? What if we had one variable for each letter of the alphabet? ★

Boolean algebra is one of the keys to the study of logic—reasoning about things which are true or false. Perhaps less obviously, but more importantly, it also fundamentally underlies computers and computer science: the ones and zeros of a computer are just T and F in disguise.[6] We'll see a bit more of this later; for now, let's begin with some fundamentals.

0.2 Statements

Usually, Boolean variables are used to represent *statements*, sentences making some sort of claim that is either true or false. For example:

(a) Australia is a continent.

[4]Or Greek letters, for that matter.[5]

[5]Or Ancient Sumerian, although I don't particularly recommend it.

[6]This would probably come as a great surprise to George Boole if he were still around to see it; although he was undoubtedly proud of his theory, he surely never imagined that it would provide a basis for much of modern technology!

(b) The Greek mathematician Euclid liked goat cheese.

(c) 16 is a prime[7] number.

(d) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ whenever $n \geqslant 1$.

Statement (a) is obviously true.[8] Statement (b) is interesting, since no one really knows whether it is true or not. However, note that it certainly *is* either true or false (Euclid either liked goat cheese or he didn't[9]); so, even though we don't know which, it still counts as a statement.[10] Perhaps someday archaeologists will unearth a papyrus fragment containing Euclid's shopping list, and we can finally settle this question once and for all. Or perhaps not. At any rate, getting on to the other two statements: (c) is false (16 is divisible by 2, 4, and 8), and (d) is . . . well, what do *you* think? Is statement (d) true or false?[11]

Examples of things which do *not* count as statements include "History class is fun" (while you may have a personal opinion on this subject, you cannot say that it is absolutely true or false in a general sense), "Why is the sky blue?" (that's a question, not a statement), and "Let n be a positive multiple of 3" (that's a command, not a statement). Inhabiting the shady netherworld between legitimate and illegitimate statements is the paradoxical "This sentence is false" (try assuming that it is either true or false and see what happens; see Chapters 1 and 2 of Hofstadter (1995) for more of this sort of thing).[12]

[7] In case you've forgotten, a prime number is a positive integer which is evenly divisible by nothing other than 1 and itself: 2, 3, 5, 7, 11, 13 . . .

[8] Unless Australia has just been made up as part of a giant conspiracy. I mean, have *you* ever been to Australia? . . . Are you *sure* it was really Australia?

[9] I suppose one could argue that perhaps Euclid never tried goat cheese. This seems unlikely, what with him being Greek and all.

[10] It is quite easy to come up with mathematical statements like this, too. In fact, we'll see several later on in this book.

[11] Try trying some examples.

[12] Actually, for various technical reasons this ultimately doesn't count as a statement (sorry to disappoint), but it isn't just silly: this sort of thing has been the catalyst for a lot of important modern work in set theory (see Chapter ??).

In addition to using simple variables, we can use 'function-like' notation to represent statements with *parameters*—that is, statements about something which has to be specified in context. For example, we might use $P(n)$ to represent "n is prime." Then we could write $P(5)$ to represent "5 is prime" (a true statement), or $P(16)$ to represent "16 is prime" (a false statement), or $P(\text{beeswax})$ to represent "beeswax is prime" (definitely a false statement). However, $P(n)$ doesn't count as a statement in and of itself, since we can't say that it is true or false; its truth value depends on the value of n.

0.3 Compound statements

In English, we combine sentences or phrases using the words *and* and *or* all the time.[13] We can also use AND and OR in Boolean algebra to connect two statements into one new statement. (To help avoid confusion, I will use capital letters for the Boolean algebra version of these words and lowercase for their normal English meanings.) The uses of AND and OR are fairly intuitive and correspond closely to the way we use these words in English (with a slight caveat for OR).

$\wedge$ (AND)

The symbol $\wedge$ represents AND, or *conjunction*.[14] If A and B are Boolean variables representing statements, $A \wedge B$ (read: A AND B) is a new statement which is true only if *both* A and B are true (and false otherwise). This is intuitive, since it corresponds exactly to the way we use the word *and* in English. If someone says, "Yesterday I bought some goat cheese and visited Grandma," but they actually only bought the goat cheese, they are not telling the truth.[15]

[13] See, I just did it twice!

[14] To help you remember what $\wedge$ means, imagine someone raising an index finger in the same way that $\wedge$ points upwards: "Oh, AND one more thing . . . "

[15] As Grandma would be quick to point out.

Figure 0.1 shows a *truth table* for $\wedge$. By showing the value of $A \wedge B$ for every possible combination of truth values for A and B, it is a concise and complete way of defining the meaning of $\wedge$.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 0.1: Truth table for $\wedge$ (AND).

Problem 0.2 ($\star\star$). Create a truth table for the expression $(A \wedge B) \wedge C$. ★

Problem 0.3 ($\star$). Using a truth table, show that $A \wedge (B \wedge C) \iff (A \wedge B) \wedge C$. (The symbol $\iff$ indicates that two expressions are *logically equivalent*—think of it as $=$, but for Boolean expressions instead of numbers.)

Problem 0.3 shows that we don't have to use parentheses when writing something like $A \wedge B \wedge C$, since it doesn't matter which of the two AND operations we perform first.

Problem 0.4 ($\star$). Show that $A \wedge B \iff B \wedge A$.

Problems 0.3 and 0.4 together show that $A \wedge B \wedge C \wedge D \wedge E$ is the same as $C \wedge E \wedge B \wedge A \wedge D$ is the same as $E \wedge D \wedge A \wedge C \wedge B$ is the same as ... you get the idea.

Problem 0.5 ($\star$). Let A represent the statement "Australia is a continent," B the statement "Grass is pink," and $C(n)$ the statement "n is odd." Which of the following statements are true, and which are false? Remember that statements can only be true or false; there's no such thing as a 'partly true' statement![16]

(a) $A \wedge B$

[16] Leave the half-truths to politicians.

(b) $C(3) \land A$
(c) $A \land C(5) \land C(8)$
(d) $B \land C(197)$

Problem 0.6 ($\star\star$). What is $A \land A$?

$\lor$ (OR)

The symbol $\lor$ represents OR, or *disjunction*.[17] Again supposing that A and B represent statements, $A \lor B$ is a new statement which is false only if both A and B are false. In other words, $A \lor B$ is true whenever A *or* B (or both) are true. The truth table shown in Figure 0.2 should help make this clear.

A	B	$A \lor B$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 0.2: Truth table for $\lor$ (OR).

This definition bears remarking upon, because we often use *or* a bit differently in English. Take this sentence as an example:

> Euclid liked goat cheese, or I'm a monkey's uncle!

Clearly, what the author of this sentence means is that one or the other of the component sentences is true—*either* Euclid liked goat cheese, *or* I'm a monkey's uncle, but *not* both. As yet another example, if your friend said, "For vacation this summer I'm going to Finland or Brazil," you would probably be very surprised to learn that your friend visited

[17] Maybe $\lor$ can remind you of someone holding out their palms: "On the one hand . . . OR on the other . . . "

both countries![18] You would interpret their statement to mean that they would visit one country or the other, but not both. This sort of *or*, which excludes the possibility of both things being true, is known as *exclusive or* and is sometimes represented by the symbol $\oplus$. However, $A \vee B$ does include the possibility of A and B being simultaneously true. Both $\vee$ and $\oplus$ play important roles in Boolean algebra, but it's just as important that you don't get them confused.

Let's establish a few basic properties of $\vee$:

Problem 0.7 ($\star$). Is $A \vee B \iff B \vee A$?

Problem 0.8 ($\star$). Is $(A \vee B) \vee C \iff A \vee (B \vee C)$?

Problem 0.9 ($\star$). What is $A \vee A$?

And now for a little practice.

Problem 0.10 ($\star$). Let A represent the statement "Australia is a continent," B the statement "Grass is pink," and $C(n)$ the statement "n is odd." Which of the following statements are true, and which are false?

(a) $A \vee B$
(b) $B \vee C(998)$
(c) $C(2) \vee A$
(d) $B \vee (A \wedge C(5))$
(e) $(C(8) \wedge B) \vee (A \vee (C(33) \wedge A))$

Problem 0.11 ($\star$). Being able to translate statements from English into Boolean algebra is a very important skill, since (outside of this chapter) you will rarely be presented with problems already expressed in terms of Boolean algebra. Can you translate each of the following English statements into Boolean expressions? (See the hint at the end of the chapter for an example.) ★

(a) It is raining, and Joe is inside.
(b) 17 is odd, prime, and complex.
(c) Either g is froopy, or g is even and $g/2$ is froopy.

[18] Right after you got over your jealousy, of course.

Problems 0.12–0.15 deal with a few properties of the way $\wedge$ and $\vee$ interact with one another.

Problem 0.12 ($\star$). Continuing in the same vein as Problems 0.3 and 0.8:

(a) Is $A \wedge (B \vee C) \iff (A \wedge B) \vee C$?
(b) What can you conclude from your answer to part (a)?

Problem 0.13 ($\star\star$). Show that $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$.

Problem 0.14 ($\star\star$). Does Problem 0.13 remind you of anything? ★

Problem 0.15 ($\star\star$). Is $A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$?

0.4 Negation

The *negation* of a statement is its logical opposite. In other words, the negation of statement S is the statement which is false whenever S is true and true when S is false. We use the symbol $\neg$ to represent negation: if S represents a statement, then $\neg S$ is its negation. Usually we would read $\neg S$ as "not S," but it can also (and perhaps more naturally) be read as "S is false." Figure 0.3 shows a truth table for $\neg$; marvel at its inscrutable complexity.

A	$\neg A$
T	F
F	T

Figure 0.3: Truth table for $\neg$.

From the truth table, it should be clear that negating a negation brings you right back where you started (if it is not the case that Euclid doesn't like goat cheese, then he likes it.) Symbolically,

$$\neg(\neg S) \iff S. \tag{0.1}$$

So we're always allowed to insert or remove a double negation anywhere we want, since it doesn't change anything.

You can usually get a good idea of a statement's negation in English by sticking the phrase "It is not the case that . . . " on the front. It's clunky, but it works. For example, if A represents the statement "Australia is a continent," then $\neg A$ represents the statement "It is not the case that Australia is a continent," or more colloquially, "Australia is not a continent." Another way is to imagine someone you don't like saying the statement, and then imagine yourself disagreeing. Like this:

> Mr. Wrong: "Statement."
> You: "That's not true! ______________________!"

Whatever you imagine yourself saying in the blank is the negation of the statement in question.[19] Of course, neither of these methods is very precise, and they're not guaranteed to give correct results (especially with more complicated statements), but they can be useful in gaining intuition. Soon, though, we'll learn rules for negating different sorts of statements expressed using Boolean algebra, so you won't have to resort to such vague techniques.

Problem 0.16 ($\star$). What is the negation of the statement "The Greek mathematician Euclid liked goat cheese"?

Problem 0.17 ($\star\star$). Let P represent the statement "All prime numbers are odd." What is $\neg P$? Can you express it without that clunky phrase "It is not the case that"? (And by the way, which is true, P or $\neg P$?) ★

Problem 0.18 ($\star\star$). Let F represent the statement "n is odd, and n is prime". What is the negation of F? ★

A *contradiction* arises when it is asserted that a statement and its negation are both true.[20] Contradictions represent patently absurd or impossible situations: Euclid can't like goat cheese and dislike goat cheese at the same time; Australia can't be a continent while simultaneously not being a continent; you can't eat your cake and not eat it, too. You get the idea. Obviously, you don't want to come across a contradiction unexpectedly, since it usually means you did something wrong. However, as we will see

[19] Sometimes.

[20] Or both false. (Which, if you think about it, amounts to the same thing.)

in section 1.4, contradictions can also be a powerful weapon in our arsenal of proof techniques.

Problem 0.19 ($\star$). Simplify:

(a) $S \wedge \neg S$
(b) $S \vee \neg S$
(c) $2B \vee \neg 2B$

Problem 0.20 ($\star\star$). Recall that $\oplus$ represents *exclusive or*: $A \oplus B$ is true when *exactly one* of A and B is true. Using $\wedge$, $\vee$, and $\neg$, construct a Boolean expression which is logically equivalent to $A \oplus B$. ★

0.5 Negating compound statements

In Problem 0.18 you were asked to find the negation of a compound statement. (Did you notice?) If you worked out the answer, you probably noticed something funny: the negation of a statement involving AND is a statement involving OR! If we think about it for a minute, it's not hard to see why. The negation of $A \wedge B$ is something that tells us when $A \wedge B$ is false. As you know, $A \wedge B$ is true only if the individual statements A and B are both true; therefore, $A \wedge B$ is false if either A or B (or both) are false. Symbolically, we can write

$$\neg(A \wedge B) \iff \neg A \vee \neg B. \tag{0.2}$$

What about compound statements using OR? I think you can figure it out:

Problem 0.21 ($\star$). What is the negation of $A \vee B$? Solve the problem before reading on! ★

Are you still reading? Hopefully that means you've solved Problem 0.21. If you haven't, then you had better march right back up the page THIS MINUTE, young lady/man, and don't come back here until you've good and solved it. Do you hear me? I mean it!

While the rest of us are waiting for you to solve it, I'll tell a joke. Why was six afraid of seven? . . . What's that? You've heard that one before? Oh, well, never mind then.

You're back? Good. Hopefully you came up with the fact that

$$\neg(A \lor B) \iff \neg A \land \neg B. \tag{0.3}$$

Of course, this looks very similar to equation (0.2), except the $\land$ and $\lor$ have switched places! There's a pleasing symmetry to these two relationships; we say that AND and OR are *dual with respect to negation*. Together, equations (0.2) and (0.3) are called *De Morgan's laws*, named after the 19th century British mathematician and logician Augustus De Morgan.

Problem 0.22 ($\star\star\star$). Prove equation (0.3) using (0.2) and (0.1). That is, start with the left side of (0.3) ($\neg(A \lor B)$), and write down a series of Boolean expressions that are all logically equivalent to each other (using (0.2) and (0.1)), ending with the right side of (0.3) ($\neg A \land \neg B$). $\bigstar$

As an aside, note that it would be easy to alter your proof from Problem 0.22 slightly to instead prove (0.2) using only (0.3) and (0.1). This means that in some sense (0.3) and (0.2) are *equivalent*, since either one implies the other.

Problem 0.23 ($\star$). Make a truth table for each of the following.

(a) $\neg A \lor B$
(b) $\neg(A \land B)$
(c) $\neg A \land \neg B$
(d) $A \lor (B \land \neg C)$

Problem 0.24 ($\star\star$). Use (0.1), (0.2) and (0.3) to simplify each of the following expressions. (For now, we will say that a Boolean expression is simplified if there are no negations directly in front of parentheses, and no double negations.)

(a) $\neg(A \land \neg B)$
(b) $\neg((A \land B) \lor C)$
(c) $\neg(A \lor B \lor C)$

Problem 0.25 ($\star\star$). Find the negation of each statement. ★

(a) "??? another statement here?"
(b) "Either 27 is a froopy number, or 99 is sedentary and Tom is lying."

0.6 Quantifiers

In the next two sections, we're going to take a step back to learn about two operators called *quantifiers*, which allow us to construct more general logical statements than would otherwise be possible. Quantifiers aren't technically a part of Boolean algebra proper; rather, they are part of a more general system called *first-order logic*. However, the distinction isn't all that important for our purposes.

Consider the following statement, which we will call G:

> G: Every even number greater than or equal to 4 can be expressed as the sum of two prime numbers.

For example, $4 = 2+2$, $6 = 3+3$, $8 = 3+5$, and so on. It's certainly a valid statement, since it is either true or false.[21] But it's different than most of the other statements we've looked at so far. The others said simple things like "16 is prime." G, on the other hand, is making a statement not just about a single number or other kind of object, but about a whole bunch of numbers all at once. In one fell swoop, it represents lots of specific statements such as "8 can be expressed as the sum of two prime numbers," "10 can be expressed as the sum of two prime numbers," "464 can be expressed as the sum of two prime numbers" . . . This idea of turning particular statements into more general ones is important, and we can always express this sort of generalization using one of two special constructions called *quantifiers*. Although they have impressive-sounding names,[22] they're actually pretty intuitive.

[21] Which do you think it is?

[22] Telling all your friends that you've been studying "universal and existential quantification" is sure to get you some strange looks. I'll leave it up to you to decide whether getting strange looks is good or bad.

The universal quantifier, $\forall$

The first quantifier we will study is the *universal quantifier*, which specifies that a particular statement is true for *all possible values* of some variable. In other words, if we have a statement $A(X)$, then we can universally quantify it to get the more general statement "$A(X)$ is true, *for all values of* X." In essence, it's a way of saying "$A(X)$ *is always true*."

As an example, let's consider the statement $L(X)$, "X likes goat cheese." Applying the universal quantifier yields the new statement "X likes goat cheese, for all values of X"—or, the way we would usually say that in English, "Everyone likes goat cheese!" (probably a false statement).

Now, if you're really, really on your toes, you might notice that I slipped a little something past you in that last paragraph. When I changed the phrase "for all values of X" into the more colloquial word "everyone," I implicitly assumed that X could only represent *people*. It's a reasonable assumption, but of course you *could* interpret "X likes goat cheese, for all values of X" to mean that not only does Euclid like goat cheese, but Australia likes goat cheese, π likes goat cheese, the solar system likes goat cheese, goat cheese likes goat cheese . . . The point is that we usually have to specify what we mean by "all values of X"—in other words, what the *universe* is from which possible values of X are chosen.[23] For example, we could say "X likes goat cheese, for all people X," or "X likes goat cheese, for all famous dead mathematicians X." If the universe is not specified, then it's usually clear from the context what it is supposed to be.

Much more important than the ability to *apply* the universal quantifier, however, is the ability to *recognize* it when you see it! Here's a good rule of thumb: if you see the words *all*, *every*, or *each*, a universal quantifier is probably involved. For example, our statement G above uses the word *every*—and sure enough, if we let $S(n)$ represent the statement "n can be expressed as the sum of two prime numbers", then we get G by applying the universal quantifier to $S(n)$ (being careful to specify the universe): "$S(n)$, for all even numbers $n \geqslant 4$."

[23] It's also sometimes known as the *universe of discourse*.

The symbol $\forall$ is universally accepted as the symbol for the universal quantifier. For example, the statement "Everyone likes goat cheese" could be represented symbolically like this:

$$\forall p : L(p).$$

If you pronounce $\forall$ as "for all," it's easy to read: "For all p, $L(p)$." Actually, to be completely correct we should specify the universe by writing something like $\forall p \in P : L(p)$, where P represents the set of all people: "For all people p, $L(p)$." The symbol $\in$ represents *set membership*—more on sets and set theory in Chapter **??**.

Problem 0.26 ($\star\star$). Express each of the following statements using a universal quantifier.

(a) ??? some problems here.

The existential quantifier, $\exists$

The *existential quantifier* specifies that a particular statement is true for *at least one value* of some variable. If we have a statement $A(X)$, then we can existentially quantify it to get the statement, "*There exists an* X such that $A(X)$ is true." It's essentially a way of saying "$A(X)$ *is sometimes true.*"

Taking the same statement from our previous example, $L(X)$, we can apply the existential quantifier to obtain the more general statement "There exists an X such that X likes goat cheese." More colloquially—assuming once again that the universe for X is people—we could say, "*Someone* in the world likes goat cheese." This is, in fact, a certifiably true statement, since I happen to personally like goat cheese. Maybe you do too, or maybe not; it doesn't matter,[24] since the statement doesn't say anything about *how many* people like goat cheese, only that there is at least one.

Statements involving an existential quantifier can be harder to spot than ones involving a universal quantifier, since there aren't usually any tell-tale

[24] Sorry.

words like *any* or *every* to look for. As an example, let's take a closer look at the statement $S(n)$, "n can be expressed as the sum of two prime numbers." There's actually an existential quantifier hiding in there—can you spot it?

Problem 0.27 ($\star\star$). Spot the existential quantifier!

To make clear the existential quantifier used by $S(n)$, we could rewrite it as "There exist prime numbers p and q such that $p + q = n$." Now do you see it? If you hadn't already figured it out for yourself, don't panic—it comes with a little practice! In general, if you see a statement claiming that something "can be" done, or some other such claim that is somehow lacking in specifics, chances are good that it involves an existential quantifier.

Problem 0.28 ($\star\star$). Rewrite each of the following statements to make explicit their use of an existential quantifier.

(a) 1130 can be written as the sum of two squares.
(b) *32767* is one less than a power of two.
(c) Not all numbers are less than a million.

The symbol $\exists$ exists in order to have a concise way of expressing an existential quantifier. For example, we can use it to express the statement "someone in the world likes goat cheese" symbolically:

$$\exists p : L(p).$$

When reading such notation you should pronounce $\exists$ as "there exists," so the above would be "There exists a p such that $L(p)$."

0.7 Negating quantified statements

In section 0.5 we learned how to negate compound statements; now let's see how to negate statements which are quantified. As an example, take the statement "All Cretans are liars," which we will call C.[25] Clearly C

[25] This is the so-called *Epimenides paradox*. Epimenides was a Cretan philosopher and religious prophet who made this statement in one of his poems. In and of itself,

involves a universal quantifier: we could rewrite it as "For all Cretans c, $L(c)$" (where this time, $L(c)$ represents the statement "c is a liar"). Now, what is $\neg C$, the *negation* of C?

Problem 0.29 ($\star\star$). What is a more idiomatic way to say "It is not the case that all Cretans are liars"?

Problem 0.30 ($\star\star$). Re-express your answer to Problem 0.29 in terms of a quantifier (if you hadn't already).

Problem 0.31 ($\star\star$). How about another: what is the negation of the statement "There exists a number $n \geqslant 10$ which is equal to the product of its digits"?

As you have probably figured out by now, the negation of a statement involving a universal quantifier is a statement involving an existential quantifier! This makes sense: if it is not the case that $A(X)$ is true *for all values* of X, then there must *exist* at least one value of X for which it is false. If it is not the case that *everyone* likes goat cheese, then there must *exist* at least one person who does *not* like goat cheese. In symbols:

$$\neg\forall X : A(X) \iff \exists X : \neg A(X) \tag{0.4}$$

And in Problem 0.31, you discovered that the negation of a statement with an existential quantifier is one with a universal: if there does not exist a value of X for which a statement is true, then it is *false* for *all* values of X.

$$\neg\exists X : A(X) \iff \forall X : \neg A(X) \tag{0.5}$$

Shades of De Morgan!! It seems that $\forall$ and $\exists$ are duals under negation, too! This is actually no coincidence, since the universal quantifier acts like a big AND over the whole universe: (if *everyone* likes goat cheese, then I like goat cheese, *and* you like goat cheese, *and* Euclid likes goat cheese, *and* Bob likes goat cheese, *and* ...so on), and the existential quantifier similarly acts like a big OR (do you see why?).

there's nothing too paradoxical about it; the paradox lies in the fact that it was made by a Cretan—so if you take it literally, he was calling himself a liar. In which case he was lying, and Cretans actually aren't liars. In which case he wasn't a liar, and they actually are. In which case . . .

Of course, the statement $A(X)$ in (0.4) and (0.5) can itself involve more quantifiers, ANDs, ORs, and so on, in which case $\neg A(X)$ can be further simplified using the appropriate rules.

Problem 0.32 ($\star\star$). Find the negation of each statement.

(a) All whole numbers are squiggly.
(b) There exists a statement S which is false.
(c) There exists an integer n such that n is odd and $f(n) < 20$.
(d) For all real numbers x, there exists an integer p such that x^p is an integer.

0.8 Implications

In mathematics, one often encounters statements of the form "If A, then[26] B," which are known as *implications*. For example, "If p is a prime number and a is a positive integer, then a^p gives the same remainder as a when both are divided by p."[27] This is another type of compound statement (since it creates one new statement, "If A, then B," out of the individual statements A and B), and can be symbolically written

$$A \implies B.$$

$A \implies B$ is true as long as B is true whenever A is true. Figure 0.4 shows a truth table for $A \implies B$, which can be read as "A implies B," or "if A, then B."

Notice that $A \implies B$ is only false when A is true but B is false. This situation is known as a *counterexample*. For example, suppose your teacher

[26]By the way, don't confuse the words *than* and *then*. *Than* is a comparative word, and is properly used in phrases such as "greater than," "less than," "math is cooler than you think," and so on. *Then* expresses time or causation: "If you study math, then you will be cool," "then we decided it was time for our SECRET WEAPON," and so on. If you mix them up, than you will make the grammar police angrier then hornets. Knot too mension the speling pollise.

[27]This is actually true, and is known as Fermat's Little Theorem. We'll encounter it again in Chapter ??.

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 0.4: Truth table for $\implies$.

said, “If you kiss a spacemonkey, you will develop dyshidrotic eczema.” Then you might say, “Your statement is false, because I kiss my pet spacemonkey all the time but I do not have small, intensely itchy vesicles on my hands and feet.”[28]

It can seem somewhat non-intuitive that $A \implies B$ is true when A is false. The idea is that $A \implies B$ only says what happens when A is true; when A is false, you certainly can't say that $A \implies B$ is false (even though it may not be true in any really meaningful sense). As an example, suppose your friend says, “If it rains tomorrow, I will stay inside.” If your friend goes out to splash in the puddles during a downpour the next day, then they lied, no question about it. But suppose the next day dawns bright and sunny, and your friend goes outside. Did they lie? Well . . . no. Your friend didn't say what they would do if it *didn't* rain, so their statement was *vacuously* true.

Problem 0.33 ($\star\star$). Show that $(A \implies B) \iff (\neg A \vee B)$.

So $A \implies B$ is really just shorthand for $\neg A \vee B$; but it's useful shorthand since it reminds us of its interpretation as an *if–then* statement.

Problem 0.34 ($\star\star$). Simplify: $\neg(A \implies B)$. Does your answer make sense? ★

Problem 0.35 ($\star\star$). Express each statement symbolically, and find its negation. Feel free to invent Boolean variables to represent various parts as necessary. ★

[28] Just one small example of the myriad real-world applications of Boolean algebra.

(a) For any integer n, if n is prime, then either n is equal to 2, or n is odd.
(b) If 156 is floopy or 99 is a Goat-Cheese Number, then the Euclid–Käse Conjecture isn't true and there exists a value of n for which n is prime, n is greater than ten million, and n is krypzoid.
(c) If p is a prime number and a is a positive integer, then a^p gives the same remainder as a when both are divided by p.

Converse and inverse and contrapositive, oh my!

Continuing in the theme of learning big, impressive-sounding math words, here are three more for you!

First, if S is the implication $A \implies B$, the *converse* of S is the implication $B \implies A$ (which can also be written $A \impliedby B$).

The *inverse* of the implication $A \implies B$ is the implication $\neg A \implies \neg B$. That is, to get the inverse you simply negate both sides of the implication.

Finally, the *contrapositive*[29] of the implication $A \implies B$ is the implication $\neg B \implies \neg A$. In other words, it's the inverse of the converse.[30]

Problem 0.36 ($\star$). What is the converse of the statement "If X is a platypus, then X is a mammal"? The inverse? The contrapositive?

Problem 0.37 ($\star\star$). Suppose you know that the implication $A \implies B$ is true. What, if anything, can you conclude about its converse? (Try some examples.) ★

Problem 0.38 ($\star\star$). Suppose you know that the implication $A \implies B$ is true. What, if anything, can you conclude about its inverse?

Problem 0.39 ($\star\star$). What, if anything, can you conclude about the contrapositive of a true implication? What about the contrapositive of a false implication?

[29] Yes, it's really called that.
[30] Or the converse of the inverse, same difference.

As you hopefully already figured out by solving the above problems, an implication and its contrapositive are equivalent (they are either both true or both false), but there's not necessarily any relationship between an implication and its converse, or an implication and its inverse. These facts are important to remember for several reasons:

(a) If you're trying to prove a particular implication, you can also consider trying to prove its contrapositive. Since an implication and its contrapositive are equivalent, it amounts to the same thing, but sometimes one is easier than the other.

(b) For whatever reason, it's a very common error to confuse an implication and its converse. Don't do it! An implication and its converse are not the same at all. This may seem obvious, but it can be tempting to mix them up. More on this in the next chapter.

Problem 0.40 ($\star$). What is the converse of the inverse of the contrapositive of the inverse of the inverse of the converse of the contrapositive of the statement, "If Euclid didn't like goat cheese, then I'm a monkey's uncle"? ★

Problem 0.41 ($\star$). All snorks are whatsits. If a whatsit is green, it is either very large or shiny (or both).

(a) Is a green snork shiny?
(b) Is a small green snork shiny?
(c) Is a shiny whatsit green?
(d) What do you know about small, rough whatsits?
(e) Can a red whatsit be very large?

Check out this totally sweet diagram (Figure 0.5)! Squiggly arrows represent finding the inverse (the symbol $\sim$ is sometimes also used to denote logical negation), dotted arrows represent finding the converse, and the long diagonal double arrows (reminiscent of $\iff$) represent finding the contrapositive. It's a nifty, compact illustration of some of the symmetries inherent in Boolean algebra.

$$\begin{tikzcd}
(A \Rightarrow B) \arrow[r, leftrightsquigarrow] \arrow[d, leftrightarrow, dotted] \arrow[dr, Leftrightarrow] & (\neg A \Rightarrow \neg B) \arrow[d, leftrightarrow, dotted] \arrow[dl, Leftrightarrow] \\
(B \Rightarrow A) \arrow[r, leftrightsquigarrow] & (\neg B \Rightarrow \neg A)
\end{tikzcd}$$

Figure 0.5: A totally sweet diagram!

Logical equivalence

Two statements A and B are said to be *logically equivalent* if $A \implies B$ *and* $B \implies A$: that is, if B is true whenever A is, and vice versa. We can abbreviate this situation symbolically by writing

$$A \iff B.$$

Of course we've seen this symbol before, but now we know what it really means. $A \iff B$ can be read as "A if and only if B," and in print you will often see "if and only if" abbreviated *iff*.[31]

Problem 0.42 ($\star\star$). Make a truth table for $(A \implies B) \wedge (B \implies A)$. Is it what you expected?

Problem 0.43 ($\star\star$). How are $\iff$ and $\oplus$ related?

0.9 A diversion: statement G

So what about our statement G from before? Do you think it's true, or not? Just as a reminder, statement G went as follows:

> G: Every even number greater than or equal to 4 can be expressed as the sum of two prime numbers.

[31]So next time you see a math book that says something about "iff," you'll know not to write to the editor reporting a typo.

Let's explore this statement a bit and see what we find. First, just for fun, let's re-express G using Boolean algebra:

$$\forall n : ((n \in 2\mathbb{Z}) \wedge (n \geqslant 4)) \Longrightarrow \exists p : \exists q : P(p) \wedge P(q) \wedge (p + q = n)$$

Nifty, huh? Impress your friends! The only part that may be unfamiliar to you is $(n \in 2\mathbb{Z})$; that means "n is an even integer."

Problem 0.44 ($\star\star$). What is $\neg G$? Express your answer in English.

Problem 0.45 ($\star\star\star$). Feeling adventurous? Try finding $\neg G$ using nothing but the rules of Boolean algebra.

One of these two statements—G or $\neg$G—must be true. But which one is it? One way to gain some intuition is to work out some examples.

Problem 0.46 ($\star$). Make a table of even numbers $n \geqslant 4$, and list ways to express each n as a sum of two prime numbers. You should make your table up to at least $n = 30$ but you can certainly go higher if you want. Figure 0.6 shows how the beginning of such a table should look.

n	$p + q$
4	$2+2$
6	$3+3$
8	$3+5$
10	$5+5=3+7$

Figure 0.6: Making a table to test the statement G.

Problem 0.47 ($\star$). If you find an even number in your table that *can't* be expressed as the sum of two primes, what can you conclude about G?

Problem 0.48 ($\star\star$). If all the even numbers in your table *can* be expressed as the sum of two primes, what can you conclude about G?

I'm going to make a wild guess that you didn't find any values of n that can't be expressed as a sum of two primes. And you probably saw in making your table that the bigger n gets, the more ways there seem to be to express n as a sum of two primes, since there are more primes to choose from. So it would be a reasonable guess at this point that G is true. But this is far from being a *proof* of G! There are a whole lot of even numbers out there (infinitely many, in fact), and we can obviously never hope to check them all. If we wanted to prove the truth of G, we would have to employ some other indirect sort of method.[32] Until then, we can't rule out the possibility of *some* lonely even number sitting out there on the number line, with no two primes that add up to it. Like $32,768$, for example.

. . . Just kidding, $32,768$ can actually be written as the sum of two primes in 435 different ways! But you get the point. At any rate, enough suspense already. G is actually a very famous mathematical statement known as the Goldbach Conjecture,[33] since it was first proposed by Christian Goldbach in a letter he wrote to Euler in June 1742 (Weisstein 2006). More than 250 years have passed since then, which has given mathematicians plenty of time to prove that G is . . . well . . . erm . . . actually, no one knows yet![34] Most mathematicians suspect that the Goldbach Conjecture is true, but no one has been able to prove it. Computer programs have been written to check lots of values of n, like we did in the table above. As of February 2007, we know that if G is actually false—if there is an even number n which cannot be written as a sum of two primes—then n has to be bigger than 5×10^{17} (that's $500,000,000,000,000,000$)! All the even numbers less than that have been checked (Oliveira e Silva 2007). But that still doesn't count as a proof of G. You never know; maybe $5 \times 10^{17} + 2$ is that lonely number that can't be expressed as the sum of two primes!

Fortunately, there are many mathematical questions which are much easier to answer than the Goldbach Conjecture. But how do we go about answering them? And once we do, how can we use the tools of logic and Boolean algebra to convince other people that our answer is correct—or more im-

[32] If you're frightened by all this talk about proofs, don't worry—the next chapter should help dispel your apprehension.

[33] A *conjecture* is like a mathematical 'guess'—a statement which some mathematician guesses is true but which is yet unproven. If a conjecture is proven to be true, it becomes a theorem.

[34] I find this one of the most fascinating things about math: it doesn't take a whole lot of searching to come across questions which are easy to ask, but very difficult to answer!

portantly, to convince ourselves that our answer is correct? The answers to these questions lie close to the shimmering heart of mathematics, and it is precisely these answers which we will explore in the next chapter.

0.10 More problems

Problem 0.49 ($\star\star$). Imagine there are a number of cards set on a table in front of you. Each card has a letter on one side, and a number on the other side. You are told that the cards should adhere to the following rule: if a card has the letter 'F' on one side, it should have the number '6' on the other.

When you look at the cards on the table, this is what you see:

$$\boxed{D} \quad \boxed{4} \quad \boxed{F} \quad \boxed{6}$$

Which cards will you have to flip over in order to determine which (if any) do not adhere to the rule?

Problem 0.50 ($\star\star\star$). Prove: $((A \implies B) \land (B \implies C)) \implies (A \implies C)$. (In other words, show that this is always true for any A, B, and C.)

Problem 0.51 ($\star\star\star$). Express each of the following statements symbolically, assuming that $P(n)$ represents the statement "n is prime", and ...???

(a) There are infinitely many primes. ??? more problems here.

Problem 0.52 ($\star\star\star$). As we have seen, any Boolean function of two variables A and B can be completely specified by a truth table with four rows, one for each possible combination of values for A and B. Since the outcome in each of the four rows could be either T or F, there are $2^4 = 16$ different possible such Boolean functions. (We have already seen three: $\land$, $\lor$, and $\oplus$.) Show that for each one of these sixteen functions, it is possible to create a logically equivalent expression using only $\land$, $\lor$, and $\neg$. (For example, the function which outputs T when

A and B are both false and F otherwise can be expressed as $\neg A \wedge \neg B$.)
★

Problem 0.53 (⋆⋆⋆). The NAND function, denoted by the symbol $\bar{\wedge}$, is a Boolean function defined as the negation of AND:

$$A \,\bar{\wedge}\, B \iff \neg(A \wedge B).$$

In other words, to find $A \,\bar{\wedge}\, B$, first find $A \wedge B$, then negate the result. A truth table for $\bar{\wedge}$ is shown in Figure 0.7.

A	B	$A \,\bar{\wedge}\, B$
T	T	F
T	F	T
F	T	T
F	F	T

Figure 0.7: Truth table for $\bar{\wedge}$.

Using only $\bar{\wedge}$, construct Boolean expressions which are logically equivalent to each of $\neg A$, $A \vee B$, and $A \wedge B$. ★

Since Problem 0.52 shows that all Boolean functions can be constructed using only $\neg$, $\wedge$, and $\vee$, and Problem 0.53 in turn shows that $\neg$, $\wedge$, and $\vee$ can be constructed using only $\bar{\wedge}$, it follows that all Boolean functions can be constructed using only $\bar{\wedge}$! This actually has important implications for computers, since once an electronic component is designed which can perform the NAND operation,[35] any conceivable logic circuit could be designed using only this component; presumably this might be cheaper and easier than designing a circuit which must use many different types of components.

It turns out (as you can verify for yourself) that $\bar{\vee}$, the negation of OR, also has this property, but $\wedge$ and $\vee$, by themselves, do not.

[35] Which can actually be done with a simple pair of transistors.

Problem 0.54 ($\star\star$). Eventually a few problems involving computer logic circuits will go here, once I can figure out how to make decent logic circuit diagrams.

Hints for Chapter 0

0.1 For only two variables, there would be four ways, as shown in this table:

A	T	T	F	F
B	T	F	T	F

0.2 It should have eight rows.

0.11 A solution to part (a) might be something like this: "$R \wedge I(\text{Joe})$, where R represents the statement 'It is raining,' and $I(p)$ represents the statement 'p is inside.'"

0.14 Try replacing $\wedge$ and $\vee$ with more familiar operators.

0.17 Start with "It is not the case that all prime numbers are odd." Say this out loud to yourself and think about what else must be true if this is true.

0.18 It *isn't* "n isn't odd, and n isn't prime."

0.20 Recall (from section 0.3) the truth table for exclusive or:

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

0.21 Take a look at the truth table for $\vee$ (Figure 0.2).

0.22 Start by using equation (0.1) to insert some double negatives:

$$\neg(A \vee B) \iff \neg(\neg\neg A \vee \neg\neg B).$$

Then use equation (0.2) in reverse.

0.25 It helps a lot to translate the English statements into Boolean expressions first!

0.34 Use the equivalence from Problem 0.33.

0.35 For example, (a) can be expressed as $P(n) \implies ((n = 2) \vee O(n))$, where $P(n)$ means "n is prime" and $O(n)$ means "n is odd."

0.37 Consider the statement in Problem 0.36, as well as the following statements: "If p is a prime number other than 2, then p is odd," and "If n is prime, then n has no divisors other than 1 and n."

0.40 Try using the diagram in Figure 0.5.

0.52 Given the truth table of a particular function, think about how to construct a Boolean expression using $\wedge$, $\vee$, and $\neg$ that describes which rows result in an output of T.

0.53 For example, $\neg A$ can be expressed as $A \bar{\wedge} A$.

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