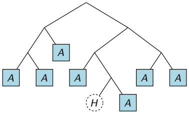
Polynomial Functors Constrained by Regular Expressions

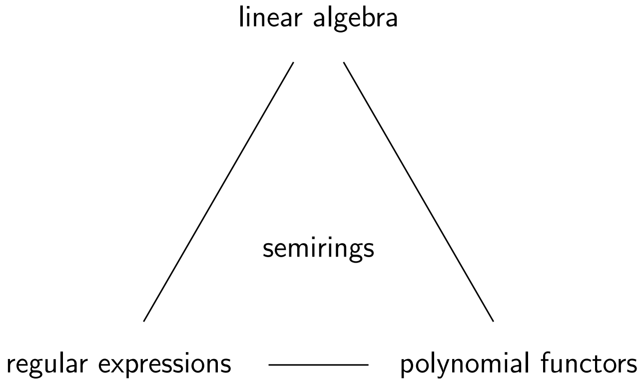


Dan Piponi Brent Yorgey

Mathematics of Program Construction
Königswinter, Germany
29 June 2015

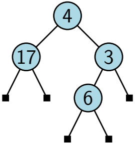
Introduction & Motivation

What this talk is about

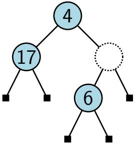


Motivation

Recall that "the **derivative** of a type is its type of one-hole contexts" (Huet, McBride, Joyal, *etc.*).



$\in T(\mathbb{N})$

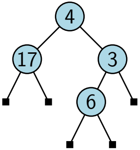


$\in \partial T(\mathbb{N})$

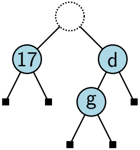
Application: **zippers**

Motivation

Recall also **dissection** (McBride, *Clowns & Jokers*).



$\in T(\mathbb{N})$



$\in \triangle\!\!\!\!| \; T(\mathbb{N}, C)$

Application: tail-recursive traversals (maps and folds)

Motivation

Questions:

- How are these related?
- Where does the definition of dissection come from?
- Where does *right* come from?

$$\mathit{right} :: F\ A + (\Delta\ F\ B\ A \times B) \cong (A \times \Delta\ F\ B\ A) + F\ B$$

Preliminaries

Polynomial functors

Polynomial functors are those functors $F : \mathbf{Set} \to \mathbf{Set}$ (or $\mathbf{Type} \to \mathbf{Type}$) inductively built from:

$$
\begin{aligned}
0(A) &= \varnothing \\
1(A) &= \{\star\} \\
X(A) &= A \\
(F + G)(A) &= F(A) \uplus G(A) \\
(F \cdot G)(A) &= F(A) \times G(A)
\end{aligned}
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Can easily generalize to multivariate polynomial functors

$$F : \mathbf{Set}^n \to \mathbf{Set}.$$

X generalizes to family of projections $X_j(A_1, \ldots, A_n) = A_j$.

Implicit/recursive definition

We also allow mutually recursive definitions:

$$
\begin{aligned}
F_1 &= \Phi_1(F_1, \ldots, F_n) \\
&\vdots \\
F_n &= \Phi_n(F_1, \ldots, F_n)
\end{aligned}
$$

interpreted as a least fixed point.

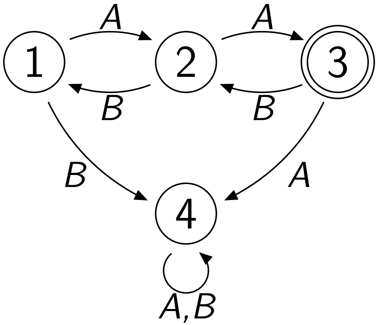
Regular expressions

Regular expressions are a language of "patterns" for strings in Σ^* (finite sequences of elements from "alphabet" Σ)

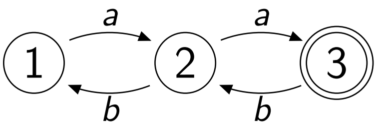
$R ::= \varnothing$	never matches
$\mid \varepsilon$	empty string
$\mid a \in \Sigma$	"a"
$\mid R_1 + R_2$	R_1 or R_2
$\mid R_1 R_2$	R_1 followed by R_2
$\mid R^*$	sequence of zero or more R

DFAs

Deterministic **F**inite **A**utomata



Deterministic **F**inite **A**utomata



Drop sink states; DFA halts and rejects if it can't take a step.

Regular expressions & DFAs

Well-known: DFAs and regular expressions are “about the same thing” (Kleene, 1951). Every regular expression has a corresponding DFA, and vice versa.

Semirings

Up to isomorphism, both polynomial functors and regular expressions form commutative **semirings** (aka **rigs**):

- Associative operations $+$, $\bullet$ with identities 0, 1
- $+$ is commutative
- $\bullet$ distributes over $+$
- $+$ does **not** necessarily have inverses (nor $\bullet$)

Other examples: $(\mathbb{N}, +, \times)$, $(\{\mathit{true}, \mathit{false}\}, \vee, \wedge)$, $(\mathbb{R} \cup \{\infty\}, \mathsf{max}, +)$

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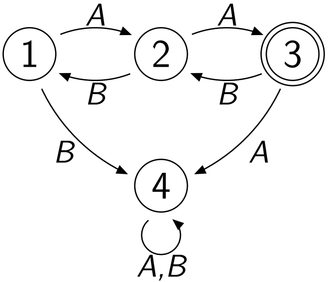
Other examples: $(\mathbb{N}, +, \times)$, $(\{\mathit{true}, \mathit{false}\}, \vee, \wedge)$, $(\mathbb{R} \cup \{\infty\}, \max, +)$

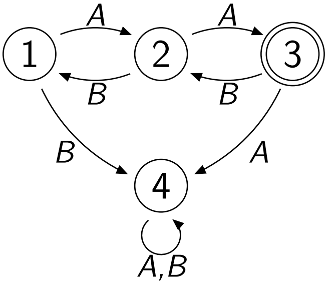
In fact, polynomial functors and regular expressions are both **star semirings**, with $x^* = 1 + x \bullet x^*$.

Transition matrices for DFAs



$$\begin{bmatrix} \cdot & A & \cdot & B \\ B & \cdot & A & \cdot \\ \cdot & B & \cdot & A \\ \cdot & \cdot & \cdot & A+B \end{bmatrix}$$

Transition matrices for DFAs



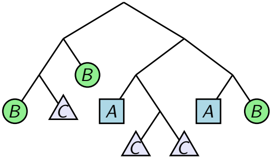
$$\begin{bmatrix} \cdot & A & \cdot & B \\ B & \cdot & A & \cdot \\ \cdot & B & \cdot & A \\ \cdot & \cdot & \cdot & A+B \end{bmatrix}$$

Interpret edge labels in an arbitrary semiring (weighted automata theory; O'Connor 2011, Dolan 2013)

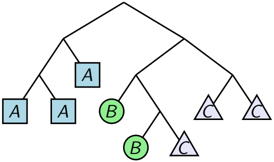
Constrained polynomial functors

Constrained polynomial functors

- Given a (univariate) F and some regular expression R over $\Sigma = \{A_1, \ldots, A_n\}$
- Want to have a restricted version F_R of F $(A_1 + \cdots + A_n)$ so the sequences of A_i (obtained from an inorder traversal) always match R.



$T(A + B + C)$



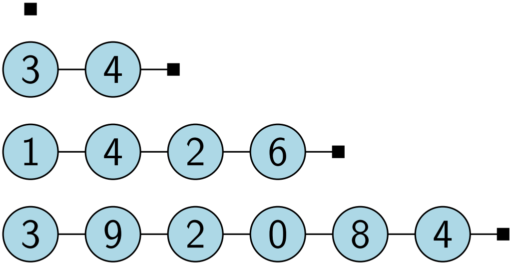
$T_{A^*B^*C^*}(A, B, C)$

Example

$$L = 1 + X \times L$$

$$R = (AA)^*$$

$L_R(\mathbb{N}) =$

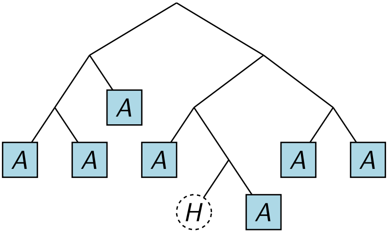


Example

$$P = X + P^2$$

$$R = A^* H A^*$$

$$P_R(A, H) \ni$$



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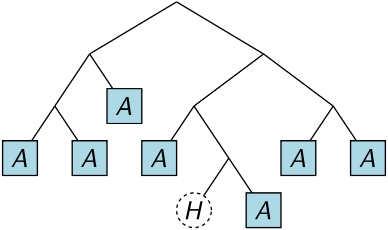
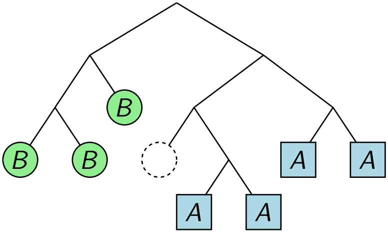


…this is differentiation! $P'(A) = P_R(A, 1)$.

Example: Dissection

$$P = X + P^2$$

$$R = B^* H A^*$$



The problem

Given a polynomial functor F and regular expression R, compute a (system of mutually recursive, multivariate) polynomial functor(s) representing F_R.

The setup

Given:

- Polynomial functor F
- DFA D

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Let F_{ij} denote the (multivariate) polynomial functor

- with same shape as F
- constrained by sequences which take the DFA from state i to state j

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Given:

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Let F_{ij} denote the (multivariate) polynomial functor

- with same shape as F
- constrained by sequences which take the DFA from state i to state j

Ultimately we are interested in $\sum_{q \in \mathrm{final}(D)} F_{1q}$.

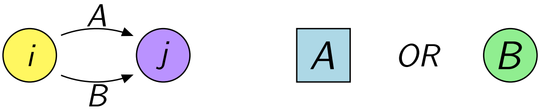
$$0_{ij} = 0$$

0 is the only thing with the same shape as 0.

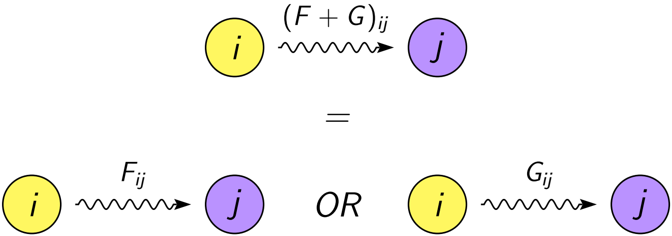
$$1_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

0 and 1 are the only things with the same shape as 1. A 1-structure doesn't make the DFA transition at all.

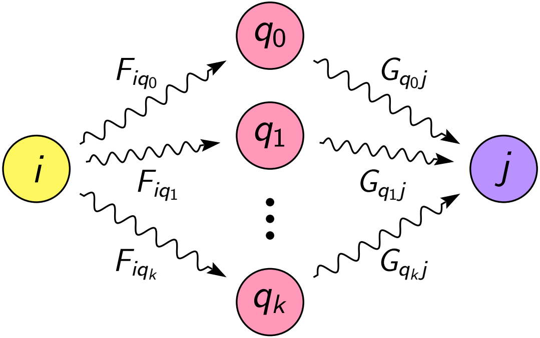
$$X_{ij} = \sum_{i \xrightarrow{A} j} X_A$$



$$(F + G)_{ij} = F_{ij} + G_{ij}$$



$$(F \bullet G)_{ij} = \sum_{q \in \mathrm{states}(D)} F_{iq} G_{qj}$$



$$
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\end{aligned}
$$

These are matrix operations! X_{ij} is the transition matrix for the DFA, interpreted in the semiring of polynomial functors.

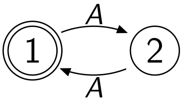
Given a DFA D,

$$F \mapsto \begin{bmatrix} F_{11} & \dots & F_{1n} \\ \vdots & \ddots & \vdots \\ F_{n1} & \dots & F_{nn} \end{bmatrix}$$

is a **semiring homomorphism** from (unary) polynomial functors to $n \times n$ matrices of (arity-$|\Sigma|$) polynomial functors.

Example

$$L = 1 + XL \qquad R = (AA)^*$$

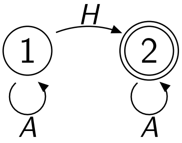


$$\begin{bmatrix} 0 & X_A \\ X_A & 0 \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & X_A \\ X_A & 0 \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$
$$= \begin{bmatrix} 1 + X_A L_{21} & X_A L_{22} \\ X_A L_{11} & 1 + X_A L_{12} \end{bmatrix}.$$

Example

$$T = 1 + XT^2 \qquad R = A^* H A^*$$

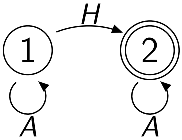


$$\begin{bmatrix} X_A & X_H \\ 0 & X_A \end{bmatrix}$$

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} X_A & X_H \\ 0 & X_A \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}^2$$
$$= \begin{bmatrix} X_A T_{11}^2 & X_A(T_{11}T_{12} + T_{12}T_{22}) + X_H T_{22}^2 \\ 0 & X_A T_{22}^2 \end{bmatrix}.$$

Derivative and dissection

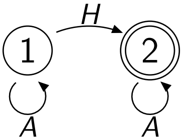
Derivative



$$\begin{bmatrix} X_A & X_H \\ 0 & X_A \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} F & F' \\ 0 & F \end{bmatrix}$$

Derivative

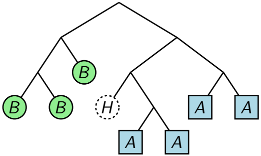


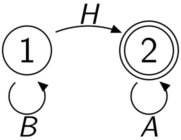
$$\begin{bmatrix} X_A & X_H \\ 0 & X_A \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} F & F' \\ 0 & F \end{bmatrix}$$

$$\begin{bmatrix} F & F' \\ 0 & F \end{bmatrix} \begin{bmatrix} G & G' \\ 0 & G \end{bmatrix} = \begin{bmatrix} FG & FG' + F'G \\ 0 & FG \end{bmatrix}$$

Dissection





$$\begin{bmatrix} X_B & X_H \\ 0 & X_A \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} \angle F & \triangle F \\ 0 & \reflectbox{$\angle$} F \end{bmatrix} \qquad \left(\begin{array}{l} \angle F \ B \ A = F \ B \\ \reflectbox{$\angle$} F \ B \ A = F \ A \end{array} \right)$$

Dissection

$$\Delta(FG) = \angle F \Delta G + \Delta F \angle G$$

Dissection

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$$\begin{bmatrix} \angle F & \Delta F \\ 0 & \text{⦣} F \end{bmatrix} \begin{bmatrix} \angle G & \Delta G \\ 0 & \text{⦣} G \end{bmatrix} = \begin{bmatrix} \angle F \angle G & \angle F \Delta G + \Delta F \text{⦣} G \\ 0 & \text{⦣} F \text{⦣} G \end{bmatrix}$$

Divided differences

$$f_{b,a} = \frac{f_b - f_a}{b - a}$$

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$f_{b,a}$ is the **average** change in f from a to b, *i.e.* the secant slope.

Note $f_{b,a} \to f'(a)$ as $b \to a$.

Divided differences and dissection?

Divided differences and dissection?

Well-known that

$$f \mapsto \begin{bmatrix} f_b & f_{b,a} \\ 0 & f_a \end{bmatrix}$$

is a semiring homomorphism.

Proof (interesting bit):

$$\begin{aligned}
(fg)_{b,a} &= \frac{(fg)_b - (fg)_a}{b - a} \\
&= \frac{(fg)_b - f_b g_a + f_b g_a - (fg)_a}{b - a} \\
&= \frac{f_b(g_b - g_a) + (f_b - f_a) g_a}{b - a} \\
&= f_b g_{b,a} + f_{b,a} g_a.
\end{aligned}$$

Divided differences and *right*

Rearranging $f_{b,a} = \frac{f_b - f_a}{b - a}$ yields

$$f_a + f_{b,a} \times b = a \times f_{b,a} + f_b$$

aka

$$\mathit{right} :: F\ A + (\Delta\ F\ B\ A \times B) \cong (A \times \Delta\ F\ B\ A) + F\ B$$

Higher-order divided differences?

$$f \mapsto \begin{bmatrix} f_c & f_{c,b} & f_{c,b,a} \\ 0 & f_b & f_{b,a} \\ 0 & 0 & f_a \end{bmatrix}$$

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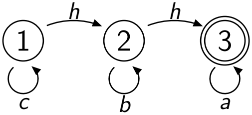

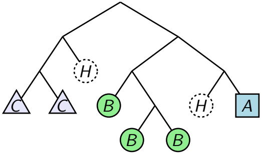

Higher-order divided differences?

$$f_{x_n \ldots x_0} = \frac{f_{x_n \ldots x_1} - f_{x_{n-1} \ldots x_0}}{x_n - x_0}.$$

Corresponding isomorphism??

Thank you!