Lecture 2: Well-orderings
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3 Well-orderings

Definition 3.1. A *binary relation* r on a set A is a subset of $A \times A$. We also define

$$\begin{aligned} \operatorname{dom}(r) &= \{ x \mid \exists y.(x, y) \in r \}, \\ \operatorname{rng}(r) &= \{ y \mid \exists x.(x, y) \in r \}, \text{and} \\ \operatorname{fld}(r) &= \operatorname{dom}(r) \cup \operatorname{rng}(r). \end{aligned}$$

Definition 3.2. A binary relation r is a *strict partial order* iff

- $\forall x.(x, x) \notin r$, and
- $\forall xyz.$ if $(x, y) \in r$ and $(y, z) \in r$ then $(x, z) \in r$,

that is, if r is irreflexive and transitive. Moreover, iff for all x and y in $\operatorname{fld}(r)$, either $(x, y) \in r$ or $(y, x) \in r$ or $x = y$, we say r is a *strict linear order.*

Definition 3.3. A strict linear order r is a *well-ordering* iff every non-empty $z \subseteq \operatorname{fld}(r)$ has an r-least member.

Remark. For example, the natural numbers $\mathbb{N}$ are well-ordered under the normal $<$ relation. However, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}^+$ are not.

However, we want to be able to talk about well-orderings longer than ω. For example,

$$0, 2, 4, \ldots, 1, 3, 5, \ldots$$

is an alternative well-ordering of the natural numbers which is longer than ω.

Definition 3.4. $f : X \to X$ is *order-preserving* iff for all $y, z \in X$, $y < z$ implies that $f(y) < f(z)$.

Theorem 3.5. *If $\langle X, < \rangle$ is a well-ordering and f is order-preserving, then for every $y \in X$, $y \leq f(y)$.*

Proof. Suppose otherwise, namely, that there exists some $z \in X$ for which $f(z) < z$. Let z_0 be the least such z. Since f is order preserving, we we also have that $f(f(z_0)) < f(z_0)$; but this contradicts the minimality of z_0. ∎

Remark. One formulation of the Axiom of Choice states that for every set x, there exists some binary relation r such that $\langle x, r \rangle$ is a well-ordering.

Theorem 3.6. *If $<$ well-orders x, then the only automorphism of $\langle x, < \rangle$ is the identity. Such a structure is called* rigid.

Proof. Let f be an automorphism (that is, an order-preserving, onto map) of $\langle x, < \rangle$. (Note that if f is order-preserving, it must be 1-1 as well.) We first note that f^{-1} is also order-preserving: if $y < z$ but $f^{-1}(y) \geq f^{-1}(z)$, we could apply f to both sides to derive a contradiction. Therefore, by Theorem 3.5, for any $y \in x$, we have $f(y) \geq y$ and $f^{-1}(y) \geq y$. Applying f to both sides of the latter inequality, we obtain $y \geq f(y)$; hence $y = f(y)$ and f is necessarily the identity. ∎

Corollary 3.7. *If $\langle x, < \rangle$ and $\langle y, <' \rangle$ are isomorphic well-orderings, there is a unique isomorphism between them. Otherwise, we could derive a non-trivial automorphism by composing one isomorphism with the inverse of another.*

Definition 3.8. Given $\langle x, < \rangle$ and $y \in x$, we can define the *initial segment of x determined by y*,

$$\mathrm{Init}(x, y, <) = \{\, z \in x \mid z < y \,\}.$$

Theorem 3.9. *If $\langle x, < \rangle$ is a well-ordering, there is no $z \in x$ for which $\langle x, < \rangle$ is isomorphic to* $\mathrm{Init}(x, z, <)$.

Remark. This is certainly *not* true for non-well-orderings. For example, $\langle \mathbb{Q}, < \rangle \cong \mathrm{Init}(\mathbb{Q}, z, <)$ for every $z \in \mathbb{Q}$!

Proof. Suppose $z \in x$ such that $\langle x, < \rangle \cong \mathrm{Init}(x, z, <)$. This is an order-preserving map that sends z to something less than itself; this contradicts Theorem 3.5. ∎

Theorem 3.10. *For every pair of well-orderings $w = \langle x, < \rangle$ and $w' = \langle y, <' \rangle$, either*

- $w \cong w'$,
- $w \cong \mathrm{Init}(w', z, <')$ *for some* $z \in y$, *or*
- $w' \cong \mathrm{Init}(w, z, <)$ *for some* $z \in x$.

Proof. Consider the set

$$f = \{\, (z, z') \mid z \in x, z' \in y, \mathrm{Init}(x, z, <) \cong \mathrm{Init}(y, z', <') \,\}.$$

We first show that f is a function. If we had (z, z') and (z, z'') both elements of f, with $z' \neq z''$, then we would have $\mathrm{Init}(y, z') \cong \mathrm{Init}(x, z) \cong \mathrm{Init}(y, z'')$. However, one of $\mathrm{Init}(y, z')$ and $\mathrm{Init}(y, z'')$ is an initial segment of the other, so this contradicts Theorem 3.9.

A similar argument shows that f is 1-1.

Note that $\mathrm{dom}(f)$ is an initial segment of $\langle x, < \rangle$, and $\mathrm{rng}(f)$ is an initial segment of $\langle y, <' \rangle$. Also note that either $\mathrm{dom}(f) = x$ or $\mathrm{rng}(f) = y$, since otherwise f could be extended. The three cases stated in the theorem correspond precisely to when both the domain and range of f are full, when the domain is full, and when the range is full. ∎