Lecture 3: Ordinals, transfinite induction
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4 Ordinals

The ordinals are canonical well-ordered sets.

Definition 4.1. A set x is *transitive* iff $\forall y. y \in x \implies y \subseteq x$.

Remark. If z is transitive, then $x \in y \in z \implies x \in z$.

Definition 4.2. x is an *ordinal* iff

- x is transitive, and
- $\langle x, \in\restriction x\rangle$ is a well-ordering.

Remark. In what follows, we use α, β, and γ to refer to arbitrary ordinals.

Lemma 4.3. *If $x \in \alpha$, then x is an ordinal.*

Proof. Since α is transitive, $x \subseteq a$; therefore it is clear that $\langle x, \in\restriction x\rangle$ is a well-ordering since $\langle \alpha, \in\restriction \alpha\rangle$ is. To see that x is transitive, suppose the contrary. That is, suppose there is some $y \in x$ and $z \in y$ such that $z \notin x$. Note that x, y, and z are all elements of α, since α is transitive. Since α is well-ordered under $\in$, either $x = z$ or $x \in z$. If $x = z$, then $z \in y \in z$, contradicting the fact that α is well-ordered; if $x \in z$, then $x \in z \in y \in x$, contradicting the fact that x is well-ordered. ∎

Lemma 4.4. *If $\beta \subseteq \alpha$ and $\beta \neq \alpha$ then $\beta \in \alpha$.*

Proof. Consider the set $\alpha - \beta$, which is nonempty by the given premises. Let γ be the $\in$-least element of $\alpha - \beta$. Then $\beta = \gamma$, which can be shown as follows.

($\subseteq$). Suppose there is some element $x \in \beta$ for which $x \notin \gamma$. Since x and γ are both elements of α, we must therefore have $\gamma \leq x \in \beta$. Since β is transitive, this implies that $\gamma \in \beta$, a contradiction.

($\supseteq$). Suppose $x \in \gamma$; then we must also have $x \in \beta$, since otherwise it would be an element of $\alpha - \beta$ less than γ, contradicting the definition of γ. ∎

Lemma 4.5. *For every α, β, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof. Suppose otherwise. Consider $\gamma = \alpha \cap \beta$, which by assumption is a proper subset of both α and β. It is easy to check that γ is an ordinal. But then by Lemma 4.4, $\gamma \in \alpha$ and $\gamma \in \beta$, so $\gamma \in \alpha \cap \beta = \gamma$, a contradiction. ∎

Theorem 4.6. *The class of ordinals is well-ordered by $\in$.*

Proof. This follows directly from Lemmas 4.4 and 4.5. ∎

Theorem 4.7. *For every set x there is an α such that $\alpha \notin x$.*

Proof. The proof of this theorem is the *Burali-Forti paradox*. Suppose there is a set x of which every ordinal is an element. Then by comprehension we may form the set

$$ord = \{ \alpha \in x \mid \alpha \text{ is an ordinal} \}.$$

But by Theorem 4.6 we can see that ord is well-ordered; by Lemma 4.3 it is transitive; hence, $ord \in ord$, a contradiction. ∎

Remark. Theorem 4.7 can equivalently be stated as "the class of ordinals is a proper class."

Some examples of ordinals:

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

can all easily be checked to be ordinals. Also, if α is an ordinal, then $\alpha \cup \{\alpha\}$ is also.

Definition 4.8. The *successor* of α, denoted $\alpha + 1$, is $\alpha \cup \{\alpha\}$.

Theorem 4.9. *$\alpha + 1$ is an ordinal. Moreover, it is the least ordinal bigger than α.*

Proof. It is easy to see that $(\alpha \cup \{\alpha\}, \in)$ is a strict linear order: for any $x, y \in \alpha \cup \{\alpha\}$, with $x \neq y$, either $x, y \in \alpha$ (in which case $x \in y$ or $y \in x$), or one of x, y is equal to α and the other is an element of α. That every non-empty subset has an $\in$-least member follows easily. To see that $\alpha \cup \{\alpha\}$ is transitive, it suffices to note that $\alpha \subseteq \alpha \cup \{\alpha\}$.

To show that $\alpha + 1$ is the least ordinal bigger than α, suppose that $\beta > \alpha$. Then by definition, $\alpha \in \beta$, and therefore $\alpha \subseteq \beta$; so $\alpha + 1 = \alpha \cup \{\alpha\} \subseteq \beta$. By Lemma 4.4, $\alpha + 1 \leq \beta$. ∎

Definition 4.10. α is a *successor ordinal* iff $\alpha = \beta + 1$ for some β. Otherwise, α is a *limit ordinal*.

Definition 4.11. The smallest non-zero limit ordinal is called ω (and it exists by the Axiom of Infinity). The elements of ω are called *natural numbers*.

Definition 4.12. $x \sim y$ iff there exists a functional relation which is a 1-1, onto mapping from x to y.

Definition 4.13. A set x is *finite* iff there exists some $n \in \omega$ for which $x \sim n$.

Theorem 4.14. *For every well-ordering $\langle x, < \rangle$ there is an ordinal α such that $\langle x, < \rangle$ is isomorphic to $\langle \alpha, \in \restriction \alpha \rangle$.*

Proof. XXX finish me! ∎

Theorem 4.15 (Transfinite Induction)**.** *If*

1. $\varphi(\emptyset)$,

2. $\varphi(\alpha) \implies \varphi(\alpha + 1)$, *and*

3. $\lim(\lambda) \wedge (\forall \beta . \beta < \lambda \implies \varphi(\beta)) \implies \varphi(\lambda)$,

then $\forall \beta . \varphi(\beta)$.

Proof. Suppose not; let γ be the $\in$-minimal ordinal for which $\neg\varphi(\gamma)$. A simple argument by cases (whether γ is $\emptyset$, a successor ordinal, or a limit ordinal) shows that γ cannot exist. $\boxtimes$