Lecture 5: Cardinals
February 2, 2009

Definition 5.7 (Well-ordering principle)**.** For every set X, there exists a bijection $f : \alpha \xrightarrow[\text{onto}]{\text{1-1}} X$ for some ordinal α.

Remark. In other words, the well-ordering principle states that every set X can be well-ordered, since there is a 1-1 projection from some ordinal onto X.

Definition 5.8 (Axiom of Choice)**.** For every set X which is a collection of nonempty sets, there exists a function f with $\text{dom}(f) = X$ and for every $y \in X$, $f(y) \in y$.

Remark. f is a "choice function" which chooses one element of each element of X.

Theorem 5.9. *The well-ordering principle and axiom of choice are equivalent.*

Proof. (WOP $\implies$ AC) Suppose X is a collection of nonempty sets. Then let $Z = \bigcup X$. By the well-ordering principle, there is some ordinal δ and some function g for which

$$Z = \{\, g(\gamma) \mid \gamma < \delta \,\}.$$

But then let $f : X \to Z$ be defined by $f(y) = g(\beta)$ where β is the least ordinal for which $g(\beta) \in y$. By definition, $\text{dom}(f) = X$ and $f(y) \in y$ for every $y \in X$. To see that f is well-defined, just note that $y \subseteq Z$ and g is onto.

(AC $\implies$ WOP) Let X be a set and define

$$Z = \mathcal{P}(X) - \{\emptyset\},$$

which is clearly a collection of nonempty sets. Let f be a choice function for Z. Then define

$$G(\beta) = f\big(X - \{\, G(\gamma) \mid \gamma < \beta \,\}\big),$$

which is clearly 1-1. Then for some δ, we have $\{\, G(\beta) \mid \beta < \delta \,\} = X$; otherwise, G would be a 1-1 function from the ordinals into X, and the ordinals would be a set (by the Replacement Axiom under G^{-1} applied to X), a contradiction.

Therefore $G \restriction \delta$ is a bijection from δ to X. ∎

Definition 5.10. The *cardinality* of X, denoted $|X|$, is the least β for which there exists an $f : \beta \xrightarrow[\text{onto}]{\text{1-1}} X$.

Remark. It is easy to see that the cardinality of any set is a cardinal (the proof is left as an exercise for the reader).

Note that we require the Axiom of Choice/Well Ordering Principle for the cardinality operator $| - |$ to be well-defined.

Theorem 5.11. *For every cardinal κ, there exists a cardinal λ with $\kappa < \lambda$.*

Proof. This follows from Cantor's theorem, since $\kappa < 2^\kappa$. ∎

Corollary 5.12. *It follows that the class of cardinals is a proper class. For if there were a set X of all cardinals, then $\bigcup X = Ord$ would be a set.*

Remark. The proof of Theorem 5.11 implicitly relied on the Axiom of Choice in its use of cardinality. We can also supply an alternative proof that does not use the Axiom of Choice:

Proof. Let κ be a cardinal and consider an ordinal $\lambda > \kappa$. If there is a 1-1 map from λ to κ, it defines a well-ordering on a subset of κ. However, the class of well-orderings on subsets of κ form a set: a well-ordering on any particular subset $z \in \mathcal{P}(\kappa)$ is just an element of the set $\mathcal{P}(z \times z)$, so by the axioms of replacement and restriction we may form the set of all such well-orderings.

Therefore, there cannot exist a 1-1 map from *every* ordinal larger than κ into κ; otherwise the ordinals would form a set.

So, choose the least ordinal for which there does not exist a 1-1 map into κ; this is the next cardinal after κ, denoted κ^+. ∎

Definition 5.13. By transfinite recursion, we define

$$\begin{aligned} \aleph_0 &= \omega \\ \aleph_{\alpha+1} &= \aleph_\alpha^+ \\ \aleph_\lambda &= \bigcup_{\beta<\lambda} \aleph_\beta \quad \text{when } \lim(\lambda). \end{aligned}$$

Remark. We note that $\aleph_\lambda$ is a cardinal: suppose there is some $f : \aleph_\lambda \xrightarrow{1\text{-}1} \gamma$, for some $\gamma < \aleph_\lambda$. Then for some $\beta < \lambda$, $\gamma < \aleph_\beta$. But then $f \restriction \aleph_{\beta+1}$ is a 1-1 function from $\aleph_{\beta+1}$ into a subset of $\aleph_\beta$, which is a contradiction by defition of $\aleph_{\beta+1}$.

Definition 5.14. $f : Ord \to Ord$ is a *normal function* iff f is order-preserving and continuous at limits (that is, $f(\lambda) = \sup_{\beta<\lambda}(f(\beta))$ for λ a limit ordinal).

Theorem 5.15. *Every normal function has arbitrarily large fixed points.*

Proof. Let f be a normal function, and pick any α. Define

$$\begin{aligned} \beta_0 &= \alpha \\ \beta_{n+1} &= f(\beta_n) \\ \beta &= \sup_{n\in\omega} \beta_n. \end{aligned}$$

Note that since f is order-preserving, $\beta_0 \leq \beta_1$. Then we have

$$\begin{aligned} f(\beta) &= \sup_{n\in\omega}(f(\beta_n)) && f \text{ is continuous} \\ &= \sup_{n\in\omega}\{\beta_{n+1}\} && \text{defn. of } \beta \\ &= \sup_{n\in\omega}\{\beta_n\} && \beta_0 \leq \beta_1 \\ &= \beta && \text{defn. of } \beta \end{aligned}$$

Hence β is a fixed point of f which is at least α. ∎

Remark. Note that $\aleph_{(-)}$ is a normal function; hence, there are arbitrarily large ordinals γ with $\gamma = \aleph_\gamma$!

Definition 5.16 (Cofinality)**.**

- $X \subseteq \alpha$ is *cofinal in* α iff $\sup(X) = \alpha$.
- A map $f : \beta \to \alpha$ is a *cofinal map* iff $\operatorname{rng} f$ is cofinal in α.
- The *cofinality of* α, denoted $\operatorname{cf}(\alpha)$, is the least β for which there exists a cofinal map $f : \beta \to \alpha$.

Remark. For example, $\operatorname{cf}(\omega) = \operatorname{cf}(\omega + \omega) = \operatorname{cf}(\aleph_\omega) = \omega$.

Note that all the fixed points constructible by the method in the proof of Theorem 5.15 have cofinality ω. This begs the question of whether there exist fixpoints with greater cofinality.

Exercise: show that if $\alpha > 0$ is a limit ordinal, then $\operatorname{cf}(\alpha)$ is a cardinal.

From now on when discussion cofinality we assume that any ordinals mentioned are nonzero limit ordinals. κ and λ will conventionally refer to cardinals.

Definition 5.17.

- κ is *regular* iff $\operatorname{cf}(\kappa) = \kappa$; otherwise it is *singular*.
- κ is a *limit cardinal* iff $\lambda < \kappa \implies \lambda^+ < \kappa$.
- κ is a *strong limit cardinal* iff $\lambda < \kappa \implies 2^\lambda < \kappa$.
- κ is *weakly inaccessible* iff it is a regular limit cardinal.
- κ is *(strongly) inaccessible* iff it is a regular strong limit cardinal,

Remark. To look ahead, we will show that if θ is strongly inaccessible, then $\langle V_\theta, \epsilon \restriction V_\theta \rangle \models \mathrm{ZFC}$.

The SI axiom asserts that there exists a strongly inaccessible cardinal; this axiom cannot be derived in ZFC.

Definition 5.18 (Cardinal arithmetic)**.**

$$\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$$
$$\kappa \times \lambda = |\kappa \times \lambda|.$$

Theorem 5.19 (Cardinal arithmetic is trivial)**.** *For all $\kappa, \lambda \geq \omega$, $\kappa \times \lambda = \kappa + \lambda = \max(\kappa, \lambda)$.*

Proof. We begin by defining a canonical map $\Gamma : Ord \times Ord \to Ord$. In particular, define $(\alpha, \beta) \prec (\gamma, \delta)$ iff either $\max(\alpha, \beta) < \max(\gamma, \delta)$, or the max's are equal and (α, β) is lexicographically smaller than (γ, δ). This defines a well-ordering on $Ord \times Ord$. Then we can define

$$\Gamma(\alpha, \beta) = \delta, \qquad \langle \delta, \epsilon \rangle \simeq \operatorname{Init}(Ord \times Ord, (\alpha, \beta), \prec).$$

We claim that $\Gamma[\kappa \times \kappa] = \kappa$ for every infinite κ, which we show by transfinite induction.

For the base case, we note that $\Gamma[\omega \times \omega] = \omega$, which is left as an exercise for the reader.

In the inductive case, let κ be the least cardinal greater than ω such that $\Gamma[\kappa \times \kappa] \neq \kappa$. Then for some $\alpha, \beta \in \kappa$, $\Gamma(\alpha, \beta) = \kappa$. Choose δ so $\max(\alpha, \beta) < \delta < \kappa$. Now, (δ, δ) determines an initial segment of $Ord \times Ord$ which contains (α, β), so $\Gamma[\delta \times \delta] \supset \kappa$, and hence $|\delta \times \delta| \geq \kappa$. However, by minimality of κ, $|\delta * \delta| = |\delta| \cdot |\delta| = |\delta| < \kappa$, a contradiction. ⊠