Lecture 6: Regularity, CH, and König's Theorem
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Lemma 5.20. *Suppose that X is a collection of sets, and that $|X| = \kappa$ and $\sup\{\, |Z| \mid Z \in X \,\} = \lambda$. Then $|\bigcup X| \leq \kappa \times \lambda$.*

Proof. By the well-ordering principle (AC), we can make an enumeration of X,

$$X = \{\, Z_\alpha \mid \alpha < \kappa \,\}.$$

For each α, $|Z_\alpha| = \lambda_\alpha < \lambda$. Again by the well-ordering principle, we can make an enumeration of each Z_α,

$$Z_\alpha = \{\, u_{\alpha\beta} \mid \beta < \lambda_\alpha\}.$$

Then we can write $\bigcup X$ as

$$\bigcup X = \{\, u_{\alpha\beta} \mid \alpha < \kappa,\, \beta < \lambda_\alpha \,\},$$

which clearly has cardinality at most $\kappa \times \lambda$. ⊠

Remark. This result is in some sense a generalization of the fact that $\mathbb{Q}$ is countable, with one important difference. To show that the rationals are countable, we just have to exhibit a bijection between the rationals (or, more simply, between $\mathbb{N} \times \mathbb{N}$) and the naturals. From this result, it seems like it should follow that if X is a countable collection of countable sets, then $\bigcup X$ is also countable; but to show this, we need the AC (which we don't need to show the countability of $\mathbb{Q}$). Intuitively, this is because we need to be able to "pick" an ordering for each $Z \in X$.

The above result is more general yet: instead of talking about a countable union of countable sets, is about a cardinality-κ union of sets with cardinality at most λ; the fact about countable sets in particular follows from the fact that $\omega \times \omega = \omega$.

Lemma 5.21. *For every ordinal α, there exists a strictly increasing cofinal map from $\mathrm{cf}(\alpha)$ to α.*

Proof. Let $g : \mathrm{cf}(\alpha) \to \alpha$ be a cofinal map. Then define $f : \mathrm{cf}(\alpha) \to \alpha$ by

$$f(\beta) = \max\{g(\beta), \sup_{\gamma<\beta}(f(\gamma) + 1)\}.$$

By definition, $\sup(\mathrm{rng}(f)) \geq \sup(\mathrm{rng}(g)) = \alpha$, so f is cofinal. f is also strictly increasing: if $\beta > \gamma$, then $f(\beta) > \sup_{\gamma<\beta} f(\gamma) \geq f(\gamma)$. ⊠

Lemma 5.22. cf *is idempotent.*

Proof. Let α, β, and γ be ordinals such that $\mathrm{cf}(\alpha) = \beta$ and $\mathrm{cf}(\beta) = \gamma$. By Lemma 5.21, suppose $f : \beta \to \alpha$ and $g : \gamma \to \beta$ are strictly increasing cofinal maps. Let $\delta \in \alpha$. Since f is a cofinal map into α, there must be some $\zeta \in \beta$ for which $f(\zeta) > \delta$. Likewise, there must be some $\eta \in \gamma$ for which $g(\eta) > \zeta$. Since f is strictly increasing, we conclude that $f(g(\eta)) > f(\zeta) > \delta$; hence $f \circ g$ is a cofinal map into α, and $\beta = \gamma$. ∎

Lemma 5.23. *If $\alpha > 0$ is a limit ordinal, then $\mathrm{cf}(\alpha)$ is an infinite, regular cardinal.*

Proof. By definition, $\mathrm{cf}(\alpha)$ is the least β for which there exists a cofinal map $f : \beta \to \alpha$ (that is, for which $\sup(\mathrm{rng}(f)) = \alpha$). Suppose that $\mathrm{cf}(\alpha)$ is not a cardinal. Then there exists some $\gamma < \mathrm{cf}(\alpha)$ such that $\gamma \sim \mathrm{cf}(\alpha)$, that is, there exists some $g : \gamma \xrightarrow[\text{onto}]{\text{1-1}} \mathrm{cf}(\alpha)$. But then $f \circ g : \gamma \to \alpha$ is also a cofinal map, contradicting the minimality of $\mathrm{cf}(\alpha)$. Also, $\mathrm{cf}(\alpha)$ must be infinite since there cannot exist a cofinal map from a finite set into an infinite one; $\mathrm{cf}(\alpha)$ is regular by Lemma 5.22. ∎

Theorem 5.24. *For every $\kappa \geq \omega$, κ^+ is regular. That is, $\aleph_{\alpha+1}$ is regular for all α.*

Remark. To help provide some intuition for the relationship of this theorem to Lemma 5.20, we can show the following special case, namely, that $\omega^+ = \aleph_1$ is regular.

Suppose otherwise, namely, that $\mathrm{cf}(\aleph_1) = \omega$ (note, by Lemma 5.23, that this is the only choice for $\mathrm{cf}(\aleph_1)$ if $\aleph_1$ is not regular). That is, for some $f : \omega \to \aleph_1$, $\mathrm{rng}(f)$ is cofinal in $\aleph_1$, *i.e.*, $\bigcup \mathrm{rng}(f) = \aleph_1$. Now, we note the following facts:

- $|\mathrm{rng}(f)| = \omega$. This is clear since $\mathrm{dom}(f) = \omega$.
- For every $\alpha \in \mathrm{rng}(f)$, $|\alpha| \leq \omega$. This follows since $\alpha \in \aleph_1$, so the biggest its cardinality could possibly be is $\aleph_0 = \omega$.

Hence $\bigcup \mathrm{rng}(f)$ is a countable union of countable sets—but we know this is countable, so it cannot be equal to $\aleph_1$.

Proof. We now give a general proof of Theorem 5.24; it follows much the same shape as the preceding remark.

For purposes of contradiction, suppose that $\aleph_{\alpha+1}$ is not regular, that is, there is some cofinal map $f : \aleph_\beta \to \aleph_{\alpha+1}$ where $\beta \leq \alpha$. Then $\bigcup \mathrm{rng}(f) = \aleph_{\alpha+1}$. If $\gamma \in \mathrm{rng}(f)$, then $|\gamma| < \aleph_{\alpha+1}$. Therefore, $|\mathrm{rng}(f)| = \aleph_\beta$ and $\sup(\mathrm{rng}(f)) = \aleph_\alpha$, so by Lemma 5.20, the cardinality of $\bigcup \mathrm{rng}(f)$ is $\alpha \times \beta = \max(\alpha, \beta) < \alpha + 1$, contradicting the cofinality of f. ∎

Remark. Theorem 5.24 asserts that all successor cardinals are regular. However, it turns out that we can't even prove that there *exist* any regular limit cardinals (*i.e.*, weakly inaccessible cardinals) other than ω!

Recall that the Continuum Hypothesis posits that there is no cardinal intermediate between ω and $|\mathcal{P}(\omega)|$; that is, there does not exist a set X such that $\omega < |X| < |\mathcal{P}(\omega)|$. Given the AC, we can reformulate this as the equality

$$2^{\aleph_0} = \aleph_1,$$

that is, $|\mathcal{P}(\omega)|$ is the next cardinal after $\aleph_0$.

This suggests what is known as the Generalized Continuum Hypothesis (GCH):

$$\forall \kappa, 2^{\kappa} = \kappa^{+}.$$

Note that in the presence of the GCH, "weakly inaccessible" and "strongly inaccessible" are equivalent.

Using the model of constructible sets, Gödel in 1939 showed that ZF + AC + GCH is consistent if ZF is; it's relatively clear what this system would look like. However, Cohen showed that ZFC + $\neg$CH is consistent if ZF is; what does ZFC look like with $\neg$CH? In fact, it turns out that for every $\alpha \geq 0$, ZFC + $(2^{\aleph_0} = \aleph_{\alpha+1})$ is consistent if ZF is! Moreover, for every λ, if $\mathrm{cf}(\lambda) > \omega$, then ZFC + $(2^{\aleph_0} = \aleph_\lambda)$ is consistent if ZF is. That is, $2^{\aleph_0}$ could be $\aleph_1$, or $\aleph_2$, or $\aleph_{\omega+1}$, but it could *not* be $\aleph_\omega$ or $\aleph_{\omega+\omega}$, and so on.

Let's prove that $\mathrm{cf}(2^{\aleph_0}) > \omega$. Strangely enough, in light of the previous remarks, this is just about all we can say about $2^{\aleph_0}$! This will follow as a corollary to Theorem 5.27.

Definition 5.25. Given a collection of sets X_i indexed by the elements of some set I, we may form the sum

$$\sum_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\}),$$

that is, the disjoint union of all the X_i's, using the indices as tags.

Definition 5.26. Given a collection of sets X_i, we may also form the product

$$\prod_{i \in I} X_i = \{\, f : I \to \bigcup_{i \in I} X_i \mid \forall i \in I,\ f(i) \in X_i \,\}.$$

That is, $\prod_i X_i$ is the set of functions which pick out an element of X_i for each $i \in I$. As an example, $\mathbb{R}^3 = \prod_{i \in \{0,1,2\}} \mathbb{R}$ is the set of functions that pick out a real number for each of the three indices 0, 1, and 2; these can also be thought of as ordered triples (although they are not actually triples in a technical sense).

Theorem 5.27 (König, 1905)**.** *Suppose that* $\forall i \in I$, $\kappa_i < \lambda_i$. *Then*

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i.$$

Remark. We defer the proof of Theorem 5.27 to examine two corollaries.

Corollary 5.28 (Cantor's Theorem (Theorem 5.5))**.**

Proof. Let $\kappa_i = 1$ and $\lambda_i = 2$. Then $\sum_{i \in I} 1 \sim I$, and $\prod_{i \in I} 2 \sim \mathcal{P}(I)$.

Corollary 5.29. $\mathrm{cf}(2^{\aleph_0}) > \omega$.

Proof. Let $f : \omega \to 2^{\aleph_0}$, and for $i \in \omega$ let $\kappa_i = |f(i)|$; thus $\kappa_i < 2^{\aleph_0}$. Also, let $\lambda_i = 2^{\aleph_0}$, for all i. Then

$$\sup_{i \in \omega} \kappa_i \le \sum_{i \in \omega} \kappa_i < \prod_{i \in \omega} \lambda_i = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}.$$

Proof. We now prove Theorem 5.27. Suppose we have a family of sets Z_i, and let $\lambda_i = |Z_i|$ and $\kappa_i < \lambda_i$ for $i \in I$. Now let $Z = \prod_{i \in I} Z_i$, and for each $i \in I$ pick (by the AC) some $Y_i \subset Z$ with $|Y_i| = \kappa_i$. Then we will show that $\bigcup_{i \in I} Y_i \ne Z$, from which the theorem follows immediately.

For each $i \in I$, define $w_i = \{\, g(i) \mid g \in Y_i \,\}$. Clearly $|w_i| \le \kappa_i < \lambda_i$. Therefore, $V_i = Z_i - w_i \ne \emptyset$, and $\prod_{i \in I} V_i \ne \emptyset$. (We note in passing that this is another formulation of the AC—that the product of a nonempty collection of nonempty sets is nonempty.)

But $\prod_{i \in I} V_i \subseteq Z$ is disjoint from $\bigcup_{i \in I} Y_i$; hence $\bigcup_{i \in I} Y_i \ne Z$.

Remark. This is a generalized "diagonal" argument, which explains why Cantor's Theorem follows so readily as a corollary. Some additional commentary should go here.