Lecture 11: The Löwenheim-Skolem Theorem
February 23, 2009

Remark. We now return to prove the Löwenheim-Skolem theorem from the previous lecture. In fact, we will prove a slightly more general version. But first, we need a lemma.

Lemma 8.9 (Tarski-Vaught $\preceq$-criterion)**.** *If $B \subseteq A$, and for all $\bar{b} \in B$ and formulas $\varphi(\bar{x}, y)$, $A \models \exists x.\varphi[\bar{b}]$ implies that there is a $b' \in B$ such that $A \models \varphi[\bar{b}, b']$, then $B \preceq A$.*

Proof. We show that

$$B \models \varphi[\bar{b}] \iff A \models \varphi[\bar{b}]$$

for all formulas φ by induction on the structure of φ. Without loss of generality, we may assume that φ does not contain $\forall$ (we can always translate $\forall$ into $\neg\exists\neg$).

- If φ is an atom, this follows from the definition of $\subseteq$ on structures.
- If $\varphi = \varphi_1 \wedge \varphi_2$, by the inductive hypothesis we know that $B \models \varphi_i[\bar{b}] \iff A \models \varphi_i[\bar{b}]$ for $i = 1, 2$. Then it is not hard to see that B satisfies $(\varphi_1 \wedge \varphi_2)[\bar{b}] = \varphi_1[\bar{b}] \wedge \varphi_2[\bar{b}]$ if and only if A does.
- The arguments for $\vee$ and $\neg$ are similar.
- If $\varphi = \exists y.\theta$. First, suppose $B \models \exists y.\theta[\bar{b}]$, and $b' \in B$ witnesses this. Then $B \models \theta[\bar{b}, b']$, which by the inductive hypothesis implies that $A \models \theta[\bar{b}, b']$, and hence that $A \models \exists y.\theta[\bar{b}]$.

  Conversely, suppose $A \models \exists y.\theta[\bar{b}]$. By assumption, there is a $b' \in B$ for which $A \models \theta[\bar{b}, b']$. But by the induction hypothesis, this shows that $B \models \theta[\bar{b}, b']$ and hence that $B \models \exists y.\theta[\bar{b}]$.

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Definition 8.10. Let A be a structure and $\varphi(\bar{x}, y)$ some formula. Then we may define a *Skolem function* f_φ for which

$$A \models \exists x.\varphi[\bar{c}, x] \implies A \models \varphi[\bar{c}, f_\varphi(\bar{c})].$$

The Skolem function f_φ picks a satisfier for the formula φ, assuming one exists.

Theorem 8.11 (Löwenheim-Skolem)**.** *If $A = \langle A, E^A \rangle$ is a structure (that is, a set with a binary relation) such that $A \models T$, then for all $X \subseteq A$, there is some B such that $X \subseteq B \subseteq A$, $B \preceq A$, and* $\mathrm{card}(B) = \aleph_0 \cdot \mathrm{card}(X)$.

Proof. First, form a set of Skolem functions

$$F = \{ f_\varphi \mid \varphi \in T \}.$$

We note that F is countable, since T is (we assume a countable language). We define X_ω, the *Skolem hull of X in A*, as follows:

$$X_0 = X$$
$$X_{i+1} = \{\, f(\bar{c}) \mid \bar{c} \in X_i \text{ and } f \in F \,\}$$
$$X_\omega = \bigcup_{i \in \omega} X_i$$

In fact, X_ω is the desired B. That $X_\omega \preceq T$ follows by construction from the Tarski-Vaught criterion. Clearly $X \subseteq X_\omega \subseteq A$. Also, $\mathrm{card}(X_\omega) \geq \aleph_0$, since it must satisfy all statements of the form "at least n elements exist"; $\mathrm{card}(X_\omega) \leq \aleph_0 \cdot \mathrm{card}(X)$, since it is a countable union of sets with size at most $\aleph_0 \cdot \mathrm{card}(X)$. ∎

Remark. We now note that the Löwenheim-Skolem Theorem is one half of a more general observation about the sizes of models.

Theorem 8.12. *For any infinite structure A and cardinal $\kappa \geq \aleph_0$, there is a structure B such that $B \equiv A$ and $\mathrm{card}(B) = \kappa$.*

Proof. Suppose $\mathrm{card}(A) = \lambda$. If $\kappa \leq \lambda$, by Theorem 8.11 we can find some $B \preceq A$ with $\mathrm{card}(B) = \kappa$, by forming the Skolem hull of some subset of A of cardinality κ; this implies that $B \equiv A$.

Conversely, suppose $\kappa > \lambda$. Let $\{\, C_\alpha \mid \alpha < \kappa \,\}$ be a set of new constant symbols. Now consider the set of formulas

$$T' = Th(A) \cup \{\, \neg(C_\alpha = C_\beta) \mid \alpha < \beta < \kappa \,\}.$$

Any finite subset of T' is satisfiable by A; hence, by compactness (Theorem 8.5), T' is satisfiable by some structure, call it B'. The cardinality of B' must be at least κ. Also, $B' \equiv A$ (with respect to the language without the extra constants C_α), and by Theorem 8.11 we may construct a $B \preceq B'$ with cardinality κ, and $B \equiv A$. ∎

Remark. For every *finite* A, on the other hand, there exists some φ_A such that $B \models \varphi_A$ if and only if $B \cong A$. That is, every finite structure can be characterized up to isomorphism in first-order logic. We can simply take φ_A to be a complete encoding of the relation on A.

Definition 8.13. A theory T is *κ-categorical* iff for all A and B, if $\mathrm{card}(A) = \mathrm{card}(B) = \kappa$ and $A \models T$ and $B \models T$, then $A \cong B$.

Remark. In other words, T is κ-categorical if it characterizes its models of cardinality κ up to isomorphism.

For example, $Th(\mathbb{Q}, <)$ is $\aleph_0$-categorical, but not $2^{\aleph_0}$-categorical, which we saw in a previous lecture.

There exist T which are not $\aleph_0$-categorical but are κ-categorical for every $\kappa > \aleph_0$. There are also trivial examples of T which are κ-categorical for all κ (for example, a set with the empty relation or total relation).

One might wonder whether there are T which are $\aleph_1$-categorical but not $\aleph_2$-categorical. The answer, as shown by M. Morley in the 1960's, is no.

Theorem 8.14 (Morley, 196?)**.** *If T is a complete, countable first-order theory, and T is κ-categorical for some $\kappa > \aleph_0$, then T is κ-categorical for all $\kappa > \aleph_0$.*

Remark. One might also wonder whether there is some countable structure A such that the second-order theory of A is not categorical? This question was shown by Ajtai to be independent of ZFC.