Lecture 13: Strongly Inaccessible Cardinals
March 2, 2009

10 Strongly inaccessible cardinals and ZF

Recall the definition of a strongly inaccessible cardinal.

Definition 10.1. κ is *strongly inaccessible* (SI) iff κ is regular and κ is a strong limit (that is, $2^\lambda < \kappa$ for every $\lambda < \kappa$). (We can also place the restriction that $\kappa > \omega$, since ω would not make a very interesting strongly inaccessible cardinal.)

Lemma 10.2. *If κ is a strongly inaccessible cardinal, then for every $\beta < \kappa$, $\mathrm{card}(V_\beta) < \kappa$.*

Proof. By induction on β. The base case ($\beta = 0$) is obvious.

Suppose $\beta = \alpha + 1$, and by the inductive hypothesis $\mathrm{card}(V_\alpha) < \kappa$. Then $\mathrm{card}(V_\beta) = 2^{\mathrm{card}(V_\alpha)} < \kappa$ since κ is a strong limit ordinal.

Now suppose β is a limit ordinal, and by the inductive hypothesis $\mathrm{card}(V_\alpha) < \kappa$ for every $\alpha < \beta$. Then $\mathrm{card}(V_\beta) = \sup_{\alpha<\beta} \mathrm{card}(V_\alpha)$, since the V_α are monotonically increasing. If this is equal to κ, then $\alpha \mapsto \mathrm{card}(V_\alpha)$ is a cofinal map $\beta \to \kappa$—but this is a contradiction, since $\beta < \kappa$ and κ is regular. ∎

Theorem 10.3. *If κ is a strongly inaccessible cardinal, then $V_\kappa \models ZF$.*

Proof. Since κ is a limit ordinal greater than ω, it is easy to see that $V_\kappa \models Z$ (that is, ZF without the Axiom of Replacement). So it only remains to show that $V_\kappa \models$ Replacement.

Let F be a functional relation, and let $x \in V_\kappa$. Then we wish to show that $F[x] \in V_\kappa$. First, note that since κ is a limit ordinal, $x \in V_\beta$ for some $\beta < \kappa$. Then since V_β is transitive, $x \subseteq V_\beta$ and hence $\mathrm{card}(x) \leq \mathrm{card}(V_\beta) < \kappa$ by Lemma 10.2.

Now let $\gamma = \sup\{\, \mathrm{rank}(F(y)) \mid y \in x \,\}$. Hence $F[x] \in V_{\gamma+1}$, so it remains only to show that $\gamma < \kappa$. But if $\gamma = \kappa$, then $y \mapsto \mathrm{rank}(F(y))$ would be a cofinal map from x to κ, a contradiction since $\mathrm{card}(x) < \kappa$. ∎

Theorem 10.4. $ZF \nvdash \exists\kappa.SI(\kappa)$.

Remark. We can show this using Gödel's second incompleteness theorem: suppose ZF could show the existence of a strongly inaccessible cardinal. Then by Theorem 10.3, we could derive $ZF \vdash \exists\kappa.V_\kappa \models ZF$. But by the completeness theorem of first-order logic, this amounts to a proof of ZF's consistency within ZF, contradicting Gödel's second incompleteness theorem.

This proof is pithy but not very illuminating. We can actually give a more elementary proof that does not rely on any incompleteness theorems. First, we'll need a lemma about strong inaccessibility.

Lemma 10.5 (Absoluteness of SI)**.** *If λ is a limit ordinal and $\kappa \in V_\lambda$, then $SI(\kappa) \iff V_\lambda \models SI(\kappa)$.*

Proof. Unfolding the definition of SI, it suffices to show each of the following.

- $\mathrm{ord}(\kappa) \iff V_\lambda \models \mathrm{ord}(\kappa)$. Since we have the Axiom of Regularity, $\mathrm{ord}(\kappa)$ simply reduces to the statement that κ is a transitive linear order, both of which are Δ_0 conditions.

- $\mathrm{card}(\kappa) \iff V_\lambda \models \mathrm{card}(\kappa)$. Recall that $\mathrm{card}(\kappa)$ holds iff there is no f for which there exists some $\beta < \kappa$ such that $f : \beta \xrightarrow[\text{onto}]{\text{1-1}} \kappa$.

  First, suppose $\mathrm{card}(\kappa)$, that is, there is no bijection in the universe between κ and some $\beta < \kappa$. If there is no such bijection in the universe, there isn't one in V_λ either, since the notion of being a bijection between β and κ is Δ_0.

  Now, suppose $V_\lambda \models \mathrm{card}(\kappa)$, and suppose by way of contradiction that there is some f in the universe which is a bijection between κ and some $\beta < \kappa$. Note that $f \subseteq \beta \times \kappa \subseteq \mathcal{P}(\mathcal{P}(\beta \cup \kappa))$, so its rank is at most two greater than the rank of κ. But $\kappa \in V_\lambda$, and since λ is a limit ordinal, $\kappa \in V_\alpha$ for some $\alpha < \lambda$, and hence $f \in V_{\alpha+2} \subseteq V_\lambda$, which is a contradiction.

- $\mathrm{cf}(\kappa) = \kappa \iff V_\lambda \models \mathrm{cf}(\kappa) = \kappa$. We can also restate $\mathrm{cf}(\kappa) = \kappa$ as the fact that there is no ordinal $\alpha < \kappa$ for which there exists a cofinal map $f : \alpha \to \kappa$.

  ($\Longrightarrow$) Suppose there is no ordinal $\alpha < \kappa$ in the universe for which there exists a cofinal map $f : \alpha \to \kappa$. Then there is no such ordinal in V_λ, either, since the notion of being a cofinal map from $\alpha \to \kappa$ is absolute for V_λ (this is because $\alpha, \kappa \in V_\lambda$; the notion of being a functional relation from α to κ is absolute for V_λ; and the predicate defining what it means to be a *cofinal* map only has to talk about union, which lowers rank).

  ($\Longleftarrow$) Suppose that $V_\lambda \models \mathrm{cf}(\kappa) = \kappa$, and suppose by way of contradiction that there is some $\alpha < \kappa$ and a cofinal map $f : \alpha \to \kappa$. Clearly $\alpha \in V_\lambda$. It is also easy to see that $f \in V_\lambda$ by the same argument as in the previous case.

- κ is a strong limit cardinal $\iff V_\lambda \models \kappa$ is a strong limit cardinal.

  First, suppose κ is a strong limit cardinal. This means that $2^\iota < \kappa$ for every cardinal $\iota < \kappa$, which is the case if and only if, for every $\iota < \kappa$, there is an injection $f : \mathcal{P}(\iota) \xrightarrow{\text{1-1}} \kappa$. By the usual rank argument, $f \in V_\lambda$.

  Now suppose $V_\lambda \models$ (κ is a strong limit cardinal), which means that for every cardinal $\iota < \kappa$, there is some $f \in V_\lambda$ such that $f : \mathcal{P}(\iota) \xrightarrow{\text{1-1}} \kappa$. But then for each ι, that f is evidence in the universe that $2^\iota < \kappa$; hence κ is a strong limit cardinal.

- κ is uncountable $\iff V_\lambda \models \kappa$ is uncountable.

  First, suppose κ is uncountable; then there does not exist any function $f : \kappa \xrightarrow{1\text{-}1} \omega$. Then in particular, there does not exist any such function in V_λ, since being an injection from κ into ω is absolute for V_λ.

  Now, suppose $V_\lambda \models \kappa$ is uncountable. By way of contradiction, suppose there is some f in the universe with $f : \kappa \xrightarrow{1\text{-}1} \omega$. By an easy rank argument (noting that κ uncountable implies $\lambda > \omega$), $f \in V_\lambda$.

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Proof of Theorem 10.4. Suppose that ZF can show the existence of a strongly inaccessible cardinal. Then there must be a smallest such cardinal λ, that is,

$$ZF \vdash \exists\lambda.(SI(\lambda) \wedge \forall\nu < \lambda.\neg SI(\nu)).$$

So $V_\lambda \models ZF$, and in particular, it must be the case that $V_\lambda \models \exists\kappa.SI(\kappa)$. So, there must be some $\kappa < \lambda$ for which $V_\lambda \models SI(\kappa)$. However, we know by Lemma 10.5 that SI is absolute for V_λ, so $SI(\kappa)$, contradicting the fact that λ is the smallest such cardinal. SDG