Lecture 14: Midterm exam review
March 16, 2009

11 Midterm exam review

Theorem 11.1 (Problem 6)**.** *For all ordinals α and β, if $\alpha < \beta$ and $V_\alpha \preceq V_\beta$, then $V_\alpha \models ZF$.*

Proof. First, we note that if $\omega < \alpha$ and $\lim(\alpha)$, then $V_\alpha \models Z$ (ZF without Replacement).

We can also easily show that α must be a limit ordinal greater than ω. First, if $\alpha = \gamma + 1$, then $\gamma \in V_\alpha$ but $\{\gamma\} \notin V_\gamma$. However, since $\alpha < \beta$, γ and $\{\gamma\}$ are both elements of V_β; this is a contradiction since $V_\alpha \preceq V_\beta$, and in particular must satisfy the formula stating that $\{\gamma\}$ exists. Second, if $\alpha = \omega$, then V_β satisfies the Axiom of Infinity but V_α does not, another contradiction.

So, it remains only to show that if α is a limit ordinal greater than ω, $\alpha < \beta$, and $V_\alpha \preceq V_\beta$, then V_α satisfies Replacement. Suppose f is a functional relation in V_α and $z \in V_\alpha$. Then for every $y \in z$, $f(y) \in V_\alpha$, and therefore $f[z] \in V_{\alpha+1} \subseteq V_\beta$. But then V_α must satisfy the formula stating that the image of z under f is a set. ∎

Remark. As an aside, we can also show that the α in the above theorem must actually be a strong limit cardinal; left as an exercise.

Remark. There was other stuff in this lecture having to do with more specific points from the exam. We also started into discussing problem 3, which is to show that ZF has no finite axiomatization. See the next lecture notes for the beginning of this.