Lecture 19: The Constructible Hierarchy, Part IV
April 1, 2009

Remark. We now proceed to prove the generalized continuum hypothesis under the assumption that $V = L$.

Lemma 13.24. *For every infinite ordinal* α, $\text{card}(L_\alpha) = \text{card}(\alpha)$.

Proof. First, we note that $L_\omega = V_\omega$, and $\text{card}(V_\omega) = \text{card}(\omega) = \omega$. Also, it is clear that $\text{card}(L_\alpha) \geq \text{card}(\alpha)$ since $\alpha \subseteq L_\alpha$.

In the successor case, we want to show that $\text{card}(L_{\alpha+1}) = \text{card}(Def(L_\alpha)) = \text{card}(\alpha+1) = \text{card}(\alpha)$. This amounts to showing that Def preserves cardinality. But every element of $Def(L_\alpha)$ is a formula together with some finite number of witnesses from L_α; hence its size is at most

$$\aleph_0 \times \sum_{n\in\omega} (\text{card}(L_\alpha))^n = \text{card}(L_\alpha).$$

Definition 13.25. $o(M)$ is the least γ for which $\gamma \notin M$. For transitive M, $o(M) = \{\gamma \mid \gamma \in M\}$.

Remark. Recall that the GCH says that $2^\kappa = \kappa^+$ for all infinite κ. To show that it holds in L, we must show that every subset of L_κ occurs at some level prior to L_{κ^+}. If we can show that $od(x) < \kappa^+$ for every $x \leq \kappa$, then $2^\kappa \leq \kappa^+$ (we already know that $2^\kappa \geq \kappa^+$ by Cantor's Theorem). In particular, we will show that for every $x \subseteq L_\alpha$, $od(x) < |\alpha|^+$.

Recall that $\alpha \mapsto L_\alpha$ is Δ_1-ZF. So there is some sentence θ for which $\alpha \mapsto L_\alpha$ is Δ_1-θ, that is, θ proves the equivalence of the Σ_1 and Π_1 forms of $\alpha \mapsto L_\alpha$. Given this, we can prove the following lemma.

Lemma 13.26. *There is a sentence* θ *such that* $\text{ZF}+(V = L) \vdash \theta$ *and for every transitive* M,

$$M \models \theta \implies \exists\alpha. \lim(\alpha) \wedge M = L_\alpha.$$

Proof. Let ψ be the function $\alpha \mapsto L_\alpha$. Then $\psi(\alpha, x)$ is absolute for transitive models of a finite fragment θ' of ZF. Then let

$$\theta = \theta' \wedge (V = L).$$

If $M \models \theta$, the claim is that $M = L_\alpha$ for some $\lim(\alpha)$.

In particular, we claim that $M = L_{o(M)}$.

- Since $\alpha \mapsto \alpha + 1$ is absolute for M, $o(M)$ must be a limit.
- $L_\alpha \subseteq M$. Since $\lim(\alpha)$, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$, and $\beta \in M$ for all $\beta < \alpha$. $\psi(\beta, x)$ is absolute for M, so $L_\beta \in M$ for all $\beta < \alpha$. Hence, $\bigcup_{\beta<\alpha} L_\beta \subseteq M$ by transitivity of M.

- $M \subseteq L_\alpha$. Note that $M \models V = L$. For $\beta < \alpha$, $(L_\beta)^M = L_\beta$; hence $M \subseteq \bigcup_{\beta<\alpha} L_\beta$.

□

Theorem 13.27. *For every x, α, if $L(x)$ and $x \subseteq L_\alpha$ then there is some $\beta < |\alpha|^+$ with $x \in L_\beta$.*

Remark. We first remark that this theorem implies the GCH; note that if $x \subseteq \kappa$ then $x \subseteq L_\kappa$. This theorem says that every subsset of L_α gets constructed at some stage prior to $|\alpha|^+$; hence the *set* of all such subsets must occur at stage $|\alpha|^+$.

Proof. Observe that θ is a conseqnece of $V = L$. Since θ is a single sentence, we can apply the Reflection Principle.

Suppose $x \subseteq L_\alpha$ and $L(x)$; hence there is some δ with $x \in L_\delta$. Pick $\beta > \delta$, $\beta > \alpha$, $\lim(\beta)$ from the club class of ordinals reflecting θ in L. Hence $x \in L_\beta$, and we note that $L_\alpha \subseteq L_\beta$.

Since AC holds in L, by the Löwenheim-Skolem theorem there is some $N \preceq L_\beta$ such that $L_\alpha \cup \{x\} \subseteq N$ and $|N| = \text{card}(\alpha)$; we also note that $N \models \theta$ since $N \preceq L_\beta$. Also, observe that $L_\alpha \cup \{x\}$ is transitive, since $x \subseteq L_\alpha$. (However, N might not be transitive.)

But N *is* extensional and well-founded, so by the Mostowski collapsing theorem, it is isomorphic to a unique transitive set M, and the isomorphism preserves $L_\alpha \cup \{x\}$ (the Mostowski isomorphism is the identity on any transitive sets).

Hence $M \models \theta$ since it is isomorphic to N. So $M = L_\gamma$, $\lim(\gamma)$, with $\alpha < \gamma < |\alpha|^+$ ($\alpha < \gamma$ since $L_\alpha \subseteq L_\gamma$; $\gamma < |\alpha|^+$ since M has cardinality α). □