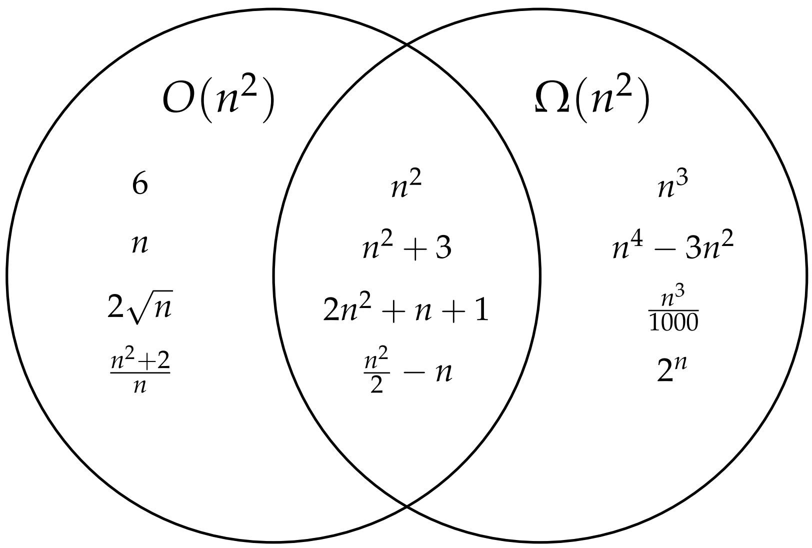
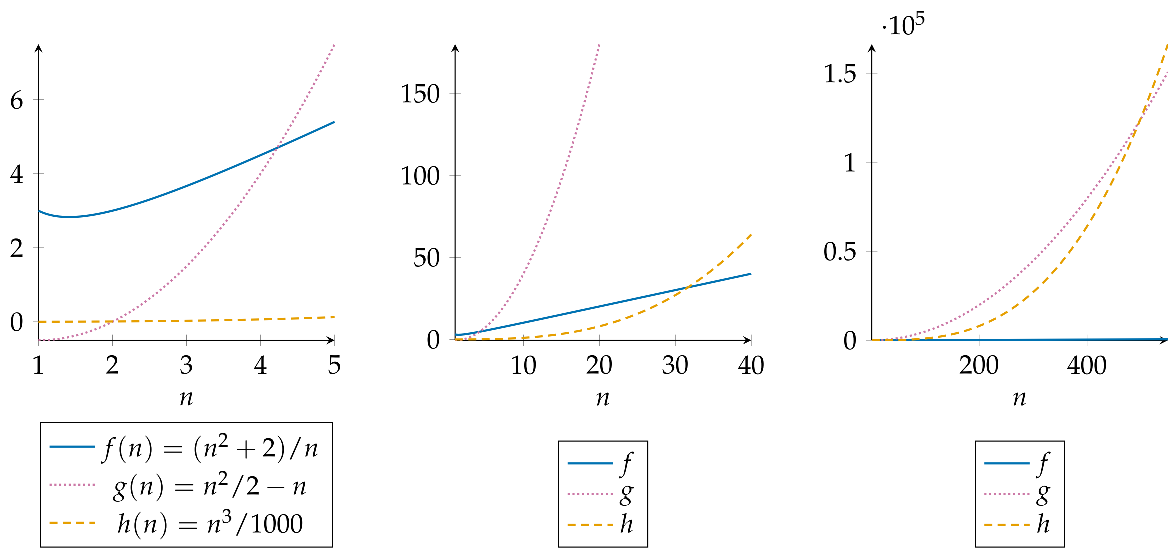
Algorithms Activity 3: Asymptotic Analysis

Model 1: Big-O and Big-Ω





Critical Thinking Questions I

Learning objective: Students will describe asymptotic behavior of functions using big-O, big-Θ, and big-Ω notation.

1 Based on the Venn diagram in the model, say whether each function is $O(n^2)$, $\Omega(n^2)$, or both.

(a) $2\sqrt{n}$

(b) n^3

(c) $2n^2 + n + 1$

(d) 2^n

Consider the functions

$$\begin{aligned} f(n) &= (n^2 + 2)/n, \\ g(n) &= n^2/2 - n, \text{and} \\ h(n) &= n^3/1000 \end{aligned}$$

for which graphs are shown in the model.

2 On each of the following intervals, list the functions f, g, and h from largest to smallest.

(a) $n \in [2, 4]$

(b) $n \in [5, 30]$

(c) $n \in [35, 450]$

3 Which function is largest, and which the smallest, at $n = 600$?

4 Does this relative order continue for all $n \geq 600$, or do the functions ever change places again? Justify your answer.

5 How do you think your answers to the previous questions relate to whether each of f, g, and h is $O(n^2)$, $\Omega(n^2)$, or both?

Say whether you think each of the following statements is true or false. Give a short justification for each answer.

6 If $f(n)$ is $O(n^2)$, then it has n^2 in its definition.

7 If $f(n)$ has n^2 in its definition, then $f(n)$ is $O(n^2)$.

8 If $f(n)$ is both $O(n^2)$ and $\Omega(n^2)$, then it has n^2 in its definition.

9 If $f(n) \leq n^2$ for all $n \geq 0$, then $f(n)$ is $O(n^2)$.

10 If $f(n)$ is $O(n^2)$, then $f(n) \leq n^2$ for all $n \geq 0$.

11 If $f(n) \leq n^2$ for all n that are sufficiently large, then $f(n)$ is $O(n^2)$.

12 If $f(n)$ is $O(n^2)$ and $g(n)$ is $\Omega(n^2)$, then $f(n) \leq g(n)$ for all $n \geq 0$.

13 Every function $f(n)$ is either $O(n^2)$ or $\Omega(n^2)$ (or both).

14 Using one or more complete English sentences and appropriate mathematical formalism, propose a correct definition of $O(n^2)$.

STOP

Critical Thinking Questions II

15 In what way(s) do you think the definition of $\Omega(n^2)$ is similar to that of $O(n^2)$?

16 In what way(s) do you think it is different?

17 Using complete English sentences, propose a definition for $\Omega(n^2)$.

18 If a function is both $O(n^2)$ and $\Omega(n^2)$, we say it is $\Theta(n^2)$. For each of the below functions, say whether you think it is $\Theta(n^2)$. Justify your answers.

(a) $3n^2 + 2n - 10$

(b) $\dfrac{n^3 - 5}{n}$

(c) $\dfrac{n^3 - 5}{\sqrt{n}}$

(d) $(n+1)(n-2)$

(e) $n + n\sqrt{n}$

19 Do you think $n^2 \cdot \log_2 n$ is $O(n^2)$, $\Omega(n^2)$, or both? Why?

Model 2: Definitions

Definition 1 (Big-O). $T(n)$ is $O(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) \leq c \cdot g(n).$$

Definition 2 (Big-Omega). $T(n)$ is $\Omega(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) \geq c \cdot g(n).$$

Definition 3 (Big-Theta). $T(n)$ is $\Theta(g(n))$ if it is both $O(g(n))$ and $\Omega(g(n))$.

Sample proof that $n^2 + 2n$ is $\Theta(n^2)$:

- First, $n^2 + 2n \leq n^2 + 2n^2 = 3n^2$ for $n \geq 1$ (since $n^2 \geq n$ when $n \geq 1$). Hence $n^2 + 2n$ is $O(n^2)$ according to the definition if we pick $c = 3$ and $n_0 = 1$.
- Next, $n^2 + 2n \geq n^2$ as long as $n \geq 0$. So by picking $c = 1$ and $n_0 = 0$, we see that $n^2 + 2n$ is also $\Omega(n^2)$.

20 Compare our class consensus definition of $O(n^2)$ with the formal definition of $O(g(n))$ above. List one way in which they are similar, and one way in which they are different.

21 Consider the following three more intuitive phrasings. Match each one with its corresponding definition.

- $T(n)$ is eventually bounded below by some constant multiple of $g(n)$.
- $T(n)$ is eventually bounded between two constant multiples of $g(n)$.
- $T(n)$ is eventually bounded above by some constant multiple of $g(n)$.

22 Which part of the definitions corresponds to the word "eventually" in Question 21?

23 In the sample proof that $n^2 + 2n$ is $O(n^2)$, the given values of c and n_0 are not the only values that would work. Given an alternate proof that $n^2 + 2n$ is $O(n^2)$ using different values of c and n_0.

24 Prove that $f(n) = 20n - 1$ is $O(n^2)$ by applying the formal definition.

25 Prove that $f(n) = n^3/10$ is $\Omega(n^2)$ by applying the formal definition.

26 Prove that $f(n) = 3n^2 - n + 1$ is $\Theta(n^2)$ by applying the formal definition.

Learning objective: Students will determine the asymptotic behavior of functions using limit theorems.

As you probably found when doing questions 23–26, it can be somewhat tedious to directly apply the formal definitions of O, Ω, and Θ. Fortunately, there is often an easier way. Consider again the functions

$$\begin{aligned} f(n) &= (n^2+2)/n, \\ g(n) &= n^2/2 - n, \text{and} \\ h(n) &= n^3/1000. \end{aligned}$$

27 What is

$$\lim_{n\to\infty} \frac{f(n)}{n^2}?$$

28 What is

$$\lim_{n\to\infty} \frac{g(n)}{n^2}?$$

29 What is

$$\lim_{n\to\infty} \frac{h(n)}{n^2}?$$

30 In general, consider the limit

$$\lim_{n\to\infty} T(n)/g(n).$$

Intuitively, what can you say about the long-term behavior of $T(n)$ relative to $g(n)$ if. . .

(a) . . . the limit exists and is equal to 0? Draw a picture.

(b) . . . the limit exists and is equal to some positive constant c? Draw a picture.

(c) . . . the limit does not exist since $T(n)/g(n)$ diverges to $+\infty$? Draw a picture.

31 Fill in the statements of the following theorems:

We will not formally prove these, although the proofs are not hard; you might like to try proving them yourself.

Theorem 4. *If*

$$0 \le \lim_{n\to\infty} \frac{T(n)}{g(n)} < \infty,$$

then $T(n)$ ____.

Theorem 5. *If*

then $T(n)$ *is* $\Omega(g(n))$.

Theorem 6. *If the limit*

$$\lim_{n\to\infty} \frac{T(n)}{g(n)}$$

exists and ___________________________, *then* $T(n)$ *is* $\Theta(g(n))$.

32 When we classify functions according to O, Θ, and Ω, we say we are describing the *asymptotic* behavior of the functions. Why do you think that word is used?

33 Describe the asymptotic behavior of

$$f(n) = 2n + \sqrt{3n} + 2$$

using big-Θ notation. Justify your answer.