Algorithms Activity 4: Asymptotic analysis II

Learning objective: Students will determine the asymptotic behavior of functions using limit theorems.

As you probably found on the previous activity, it can be somewhat tedious to directly apply the formal definitions of O, Ω, and Θ. Fortunately, there is often an easier way. Consider again the functions

$$\begin{aligned} f(n) &= (n^2 + 2)/n, \\ g(n) &= n^2/2 - n, \text{and} \\ h(n) &= n^3/1000. \end{aligned}$$

1 (Review) Say whether each of f, g, and h is $O(n^2)$ only, $\Omega(n^2)$ only, or $\Theta(n^2)$ (*i.e.* both).

2 What is

$$\lim_{n\to\infty} \frac{f(n)}{n^2}?$$

3 What is

$$\lim_{n\to\infty} \frac{g(n)}{n^2}?$$

4 What is

$$\lim_{n\to\infty} \frac{h(n)}{n^2}?$$

5 In general, consider the limit

$$\lim_{n\to\infty} T(n)/g(n).$$

Intuitively, what can you say about the long-term behavior of $T(n)$ relative to $g(n)$ if...

(a) ...the limit exists and is equal to 0? Draw a picture.

(b) ...the limit exists and is equal to some positive constant c? Draw a picture.

(c) ...the limit does not exist since $T(n)/g(n)$ diverges to $+\infty$? Draw a picture.

6 Fill in the statements of the following theorems:

We will not formally prove these, although the proofs are not hard; you might like to try proving them yourself, based on the formal definitions of O and Ω.

Theorem 1. *If*

$$0 \leq \lim_{n\to\infty} \frac{T(n)}{g(n)} < \infty,$$

then $T(n)$ ____.

Optional challenge problem to think about later: why do these theorems say "if" and not "if and only if"? *Hint*: consider a function like

$$f(n) = \begin{cases} n^2 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}.$$

Theorem 2. *If*

then $T(n)$ *is* $\Omega(g(n))$.

Theorem 3. *If the limit*

$$\lim_{n\to\infty} \frac{T(n)}{g(n)}$$

exists and __________________________, *then* $T(n)$ *is* $\Theta(g(n))$.

7 When we classify functions according to O, Θ, and Ω, we say we are describing the *asymptotic* behavior of the functions. Why do you think that word is used?

8 Describe the asymptotic behavior of

$$f(n) = 2n + \sqrt{3n} + 2$$

using big-Θ notation. Justify your answer.

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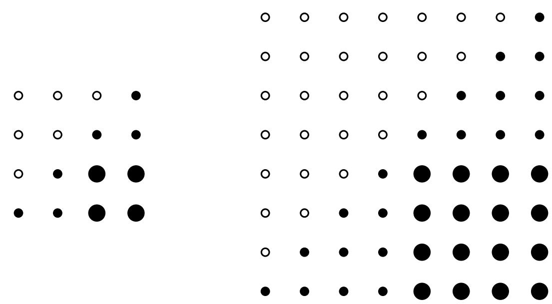
Model 1: Three proofs



$$
\begin{array}{ccccccccccc}
1 & + & 2 & + & 3 & + & \dots & + & (n-1) & + & n \\
n & + & (n-1) & + & (n-2) & + & \dots & + & 2 & + & 1 \\
\hline
(n+1) & + & (n+1) & + & (n+1) & + & \dots & + & (n+1) & + & (n+1)
\end{array}
$$

$$
\begin{aligned}
1+2+\cdots+n &< n+n+\cdots+n && (1)\\
&= n^2 && (2)
\end{aligned}
$$

$$
\begin{aligned}
1+2+\cdots+n &> n/2+(n/2+1)+\cdots+n && (3)\\
&> n/2+n/2+\cdots+n/2 && (4)\\
&= (n/2)^2 && (5)\\
&= n^2/4 && (6)
\end{aligned}
$$

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Learning objective: Students will understand and prove the asymptotic behavior of $1 + 2 + 3 + \cdots + n$ and $1 + 2 + 4 + 8 + \cdots + 2^n$.

Learning objective: Students will apply geometric, algebraic, and inequational reasoning to asymptotic behavior.

The first row of Model 1 actually shows two similar diagrams at different sizes, one 4×4 and one 8×8. Each diagram consists of a bunch of dots—some hollow and some filled; and the filled dots come in two varieties, big and small.

9 How many dots are there in total in the first diagram? In the second diagram?

10 How many big dots are there (*i.e.* the lower-right square) in the first diagram? How many are in the second?

11 How many filled dots are there in total (both big and small filled dots, *i.e.* the lower-right triangle) in the first diagram? In the second?

Now suppose that we abstract away the specific sizes of the diagrams and imagine a generic $n \times n$ version of the same diagram. To make things slightly simpler, assume that n is even.

12 In terms of n, how many dots would there be in total?

13 In terms of n, how many big dots would there be in the lower right?

14 Explain why the number of filled-in dots is equal to

$$1 + 2 + 3 + \cdots + n.$$

15 Based on the diagrams, what can you say about the relationship between these three quantities?

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16 In terms of concepts you explored on the previous activity, what does this prove about the sum $1+2+3+\cdots+n$?

Now consider the second proof.

17 Notice that the top row is our friend $1+2+3+\cdots+n$. What is the second row?

18 Why does the bottom row consist of copies of $(n+1)$?

19 What is the sum of the bottom row?

20 Use this to derive a formula for $1+2+\cdots+n$ in terms of n.

21 What does this formula imply about the asymptotic behavior of $1+2+\cdots+n$? Justify your answer.

Finally, consider the third proof. Surprise!—once again it has to do with the sum $1+2+3+\cdots+n$. For this proof we will again assume n is even.[1]

[1] It is not hard to fix the proof to work for odd n as well, but the details would end up obscuring the main idea somewhat.

22 Why is step (1) true?

23 Why is the right-hand side of (1) equal to (2) ?

24 What does this prove about $1+2+\cdots+n$?

25 Now, what is happening in step (3)?

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26 Why is the right-hand side of (3) greater than (4)?

27 Why is (4) equal to (5)?

28 What does this prove about $1 + 2 + \cdots + n$?

29 Two of these three proofs are in some sense the same. Which two?

30 Which proof do you think you will remember best? Why? (If different members of your group would answer this question differently, just note the different answers and the reasons.)

31 Using whatever method you wish, prove that

$$1 + 2 + 4 + 8 + \cdots + 2^n \text{ is } \Theta(2^n).$$

32 What method did you use for the previous question? Explain why you chose it.

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