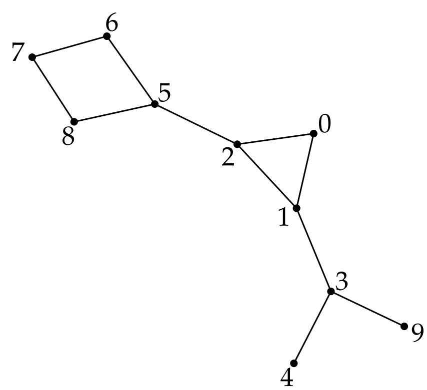
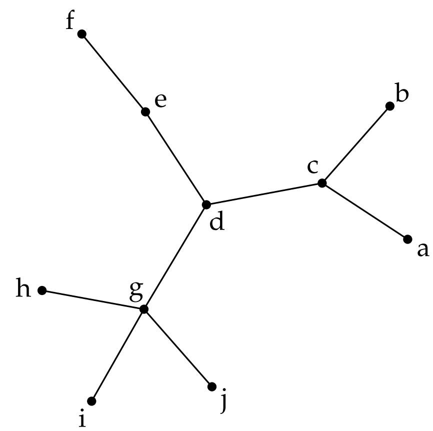
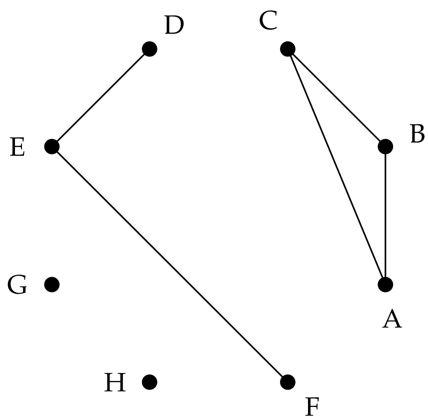
Algorithms Activity 5: Graphs

Model 1: Graphs (review)







$G = (V, E)$
$V = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta\}$
$E = \{\{\alpha, \delta\}, \{\theta, \eta\}, \{\beta, \alpha\}, \{\zeta, \delta\}, \{\epsilon, \eta\}, \{\gamma, \alpha\}\}$

Definition 1. A *graph* $G = (V, E)$ is a set of *vertices* V together with a set E of *edges*, where each edge consists of a set of two vertices.

Above are shown four example graphs.

Learning objective: Students will understand and apply graph terms *edge*, *vertex*, *adjacent/neighbor*, *degree*, *leaf*, *path*, *connected*, *connected component*, *cycle*, *cyclic*, *acyclic*, and *tree*.

For each of the following terms, state its definition, and give one example from the model and, if appropriate, one non-example from the model. If no one in your group can remember the definition of a term, give it your best guess.

1 *vertex*[1]

[1] Vertices are also called *nodes*.

2 *edge*

3 *adjacent* vertices[2]

[2] Adjacent vertices are also referred to as *neighbors*.

4 *degree* of a vertex

5 *leaf*

6 *path*

7 *connected* vertices (what does it mean for *two* vertices to be connected?)

8 *connected* graph (what does it mean for an entire graph to be connected?)

9 *disconnected* graph

10 *connected component*

11 *cycle*

12 *cyclic* graph

13 *acyclic* graph

A few more questions for you to ponder:

14 Suppose we draw an edge from a vertex back to itself. Does the given definition of a graph allow this?

Sometimes it makes sense to allow these things and sometimes it doesn't; you just have to be clear about what kind of graph you have.

15 Suppose we draw two edges between the same pair of vertices. Does the given definition of a graph allow this?

16 How many vertices can be in a cycle?

17 The lowercase graph is a *tree*. The number graph and uppercase graph are not trees. What do you think is the definition of a tree?

Warning—a tree graph is not quite the same thing as a tree data structure!

18 Is the Greek graph a tree?

Some proofs about graphs

Learning objective: Students will write proofs about graphs.

Theorem 2 (Trees). *Let $G = (V, E)$ be a graph with $|V| = n \geq 1$. Any two of the following imply the third:*

1. *G is connected.*
2. *G is acyclic.*
3. *G has $n - 1$ edges.*

We will take each pair of statements in turn and show that they imply the third. Fill in the blanks to complete the following proofs! Note that the size of a blank does not necessarily correspond to the amount of stuff you should write in it.

Lemma 3. $(1), (2) \implies (3)$. *That is: let $G = (V, E)$ be a graph with*

$|V| = n \geq 1$. If ________________________________

and ________________________________,

then ________________________________.

Proof. Let $P(n)$ denote the statement "Any graph G with n vertices

which is ________________ and ________________

must have ________________________________."

We wish to show that $P(n)$ holds for all $n \geq 1$.

The proof is by ________________________________.

- The base case is when ________________________________.

  In this case, G must be ________________________________

  which indeed ________________________________.

- In the inductive case, suppose $P(k)$ holds for some $k \geq 1$. That is,

  suppose that any graph with ________________ vertices

  which is ________________________________

  must have ________________________________.

  Then we wish to show ________________________________.

So, let G be a graph with ____________________ vertices which is

_________________________ and _________________________.
We claim that G must have some vertex which is a leaf, that is, a

vertex of degree ____:

- G cannot have any vertices of degree ______________________

  because ____.

- It also cannot be the case that every vertex of G has degree $\geq$ __.

  If they did, then we could find a ___________ by starting at any

  vertex and walking along edges randomly until _____________;

  we would never get stuck because ________________________.

  However, this is impossible because we assumed ____________.

Hence, G must have some vertex which ______________________.
If we delete this vertex along with the edge adjacent to it, it results

in a graph G' with only ____________________________ vertices;

we note that G' is still ____________________________________

because ____

and also ____

because ____.
Hence we may apply the inductive hypothesis to conclude that G'

________________________. Adding the deleted vertex and edge

back to G' shows that G ___________________________________,
which is what we wanted to show.

$\square$

Let's do one more! (You will do the third on your HW.)

Lemma 4. $(2), (3) \implies (1)$, *that is,* ______________________________

____.

Proof. This proof uses a *counting argument*: we will show what we wish to show by counting things in multiple ways.

Let c denote the number of connected components of G. We want to show that ________________________________.

Number the components of G from $1 \dots c$, and say that component i has n_i vertices. Then

$$\sum_{i=1}^{c} n_i = \underline{\hspace{4cm}}$$

because ____.

Each connected component is by definition a ______________ graph; each component must also be ________________________ since we assumed that G is. Hence we may apply Lemma **??** to conclude that component i ______________________________.

Adding these up, the total number of edges in G is

$$|E| = \sum_{i=1}^{c} \underline{\hspace{2.5cm}} = \underline{\hspace{5cm}}$$

But we already assumed the number of edges in G is ____________, and hence ______________ as desired. □