Induction exemplars

CSCI 382, Algorithms

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Weak induction

Theorem 0.1. *For all $n \geq 1$, the sum of the first n odd numbers is equal to n^2, that is,*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

For example, $1 + 3 + 5 + 7 = 16 = 4^2$. Let's give the statement "$1 + 3 + 5 + \cdots + (2n - 1) = n^2$" a name: we'll call it $P(n)$. We want to show that $P(n)$ is true for all $n \geq 1$. The idea is to use induction to do this by showing

$$P(1) \implies P(2) \implies P(3) \implies P(4) \implies \ldots,$$

that is, we first show $P(1)$ is true (the base case), and then we show that each $P(k)$ implies $P(k+1)$ (the inductive step). Just like dominoes, this "sets off a chain reaction" that proves $P(n)$ for all $n \geq 1$.

Proof. By (weak) induction on n.

- In the base case, when $n = 1$, the sum is just 1, and $n^2 = 1^2 = 1$ as well.
- In the inductive case, suppose $P(k)$ is true for some particular $k \geq 1$, that is, suppose

  $$1 + 3 + 5 + \cdots + (2k - 1) = k^2. \qquad \text{(IH)}$$

  IH is our *induction hypothesis*.

  We must then show that $P(k+1)$ is also true, that is, that $1 + 3 + 5 + \cdots + (2(k+1) - 1) = (k+1)^2$. We can show this as follows:

  $$\begin{aligned} & 1 + 3 + 5 + \cdots + (2(k+1) - 1) \\ = & \qquad \{ \quad \text{algebra} \quad \} \\ & 1 + 3 + 5 + \cdots + (2k+1) \\ = & \qquad \{ \quad \text{listing one more term that was part of the } \ldots \quad \} \\ & 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\ = & \qquad \{ \quad \text{IH} \quad \} \\ & k^2 + (2k+1) \\ = & \qquad \{ \quad \text{factor} \quad \} \\ & (k+1)^2. \end{aligned}$$

  Notice how we make use of our assumption (the induction hypothesis) to rewrite $1 + \cdots + (2k-1)$ into k^2.

So, we have shown that $P(1)$ is true, and that whenever $P(k)$ is true then $P(k+1)$ is also true. Hence, by induction, $P(n)$ is true for all $n \geq 1$. ∎

Practice

Theorem 0.2. *Let the Fibonacci numbers be defined as usual, with $F_0 = 0$ and $F_1 = 1$. Then for all $n \geq 0$,*

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

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FOO(n) =
  if n = 0
    then 0
    else 2 × FOO(n − 1) + 1
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Theorem 0.3. *For all $n \geq 0$, $FOO(n) = 2^n - 1$.*

Strong induction

Theorem 0.4. *For all $n \geq 0$, $F_n \geq 0$.*

This theorem is a bit silly—it seems rather obviously true—but it's worthwhile seeing how to formally prove it by induction. Again, let us give the name $Q(n)$ to the statement "$F_n \geq 0$"; we want to prove that $Q(n)$ holds for all $n \geq 0$. However, this time we will not be able to prove

$$Q(0) \implies Q(1) \implies Q(2) \implies \dots$$

as we did in the previous proof. The reason is that $Q(k)$ by itself is not enough to imply $Q(k+1)$: since each Fibonacci number is defined in terms of the *two* previous Fibonacci numbers, we need to know both $Q(k-1)$ and $Q(k)$ in order to conclude $Q(k+1)$. That is, if all we know is that $F_k \geq 0$, we can't say for sure that $F_{k+1} \geq 0$: what if F_{k-1} is negative? But if we know that F_{k-1} and F_k are both nonnegative, then F_k must be as well since it is a sum of nonnegative things.

So, instead of using weak induction, where we suppose $Q(k)$ and use it to prove $Q(k+1)$, we will use *strong induction*, where we suppose that *all* the $Q(j)$ from $Q(0)$ up to some $Q(k)$ are true, and use them to show $Q(k+1)$.

Proof. By (strong) induction on n.

- In the base cases, when $n = 0$, we have $F_0 = 0 \geq 0$, and when $n = 1$, we have $F_1 = 1 \geq 0$.
- In the inductive case, suppose that for some $k \geq 0$, we know that $Q(j)$ is true for *all* $0 \leq j \leq k$. We must show that $Q(k+1)$ is true, that is, that $F_{k+1} \geq 0$.

Notice that we need *two* base cases, since the inductive step needs to refer back to two previous Fibonacci numbers. Without two base cases to start, the first inductive step would never get off the ground.

$$
\begin{array}{lll}
 & F_{k+1} & \\
= & & \{ \quad \text{definition} \quad \} \\
 & F_k + F_{k-1} & \\
\geq & & \{ \quad \text{IH twice, with } j = k \text{ and } j = k-1 \quad \} \\
 & 0 + 0 & \\
= & & \{ \quad \text{arithmetic} \quad \} \\
 & 0 &
\end{array}
$$

SDG

Practice

Theorem 0.5. *For all $n \geq 0$, $F_n \leq 2^{n-1}$.*