Algorithms: Asymptotic analysis II: limit theorems

Learning objective: Students will determine the asymptotic behavior of functions using limit theorems.

As you probably found on the previous activity, it can be somewhat tedious to directly apply the formal definitions of O, Ω, and Θ. Fortunately, there is often an easier way. Consider again the functions

$$\begin{aligned} f(n) &= (n^2+2)/n, \\ g(n) &= n^2/2 - n, \text{and} \\ h(n) &= n^3/1000. \end{aligned}$$

1 (Review) Say whether each of f, g, and h is $O(n^2)$ only, $\Omega(n^2)$ only, or $\Theta(n^2)$ (*i.e.* both).

2 What is

$$\lim_{n\to\infty} \frac{f(n)}{n^2}?$$

3 What is

$$\lim_{n\to\infty} \frac{g(n)}{n^2}?$$

4 What is

$$\lim_{n\to\infty} \frac{h(n)}{n^2}?$$

5 In general, consider the limit

$$\lim_{n\to\infty} T(n)/g(n).$$

Intuitively, what can you say about the long-term behavior of $T(n)$ relative to $g(n)$ if...

(a) ...the limit exists and is equal to 0? Draw a picture.

(b) ...the limit exists and is equal to some positive constant c? Draw a picture.

(c) ...the limit does not exist since $T(n)/g(n)$ diverges to $+\infty$?
Draw a picture.

6 Fill in the statements of the following theorems:

We will not formally prove these, although the proofs are not hard; you might like to try proving them yourself, based on the formal definitions of O and Ω.

Theorem 1. *If*

$$0 \le \lim_{n\to\infty} \frac{T(n)}{g(n)} < \infty,$$

then $T(n)$ ____.

Optional challenge problem to think about later: why do these theorems say "if" and not "if and only if"? *Hint*: consider a function like

$$f(n) = \begin{cases} n^2 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}.$$

Theorem 2. *If*

then $T(n)$ *is* $\Omega(g(n))$.

Theorem 3. *If the limit*

$$\lim_{n\to\infty} \frac{T(n)}{g(n)}$$

exists and ____________________________, *then* $T(n)$ *is* $\Theta(g(n))$.

7 Describe the asymptotic behavior of

$$f(n) = 2n + \sqrt{3n} + 2$$

using big-Θ notation. Justify your answer.