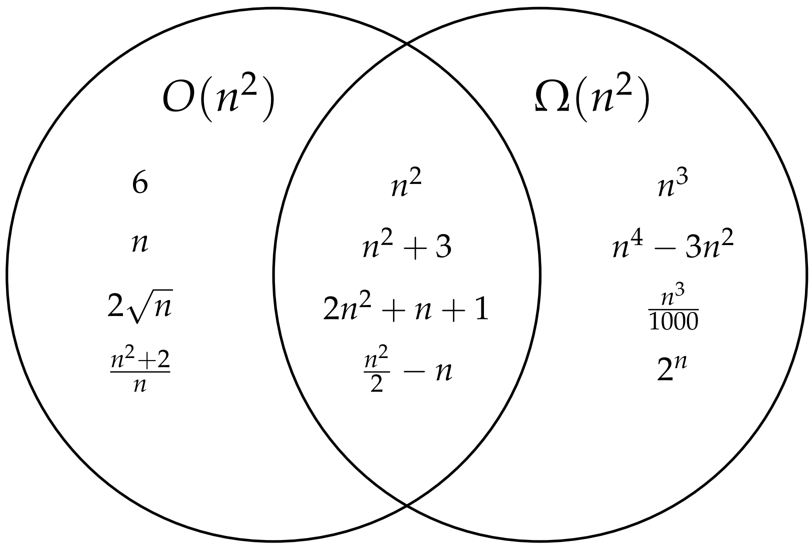
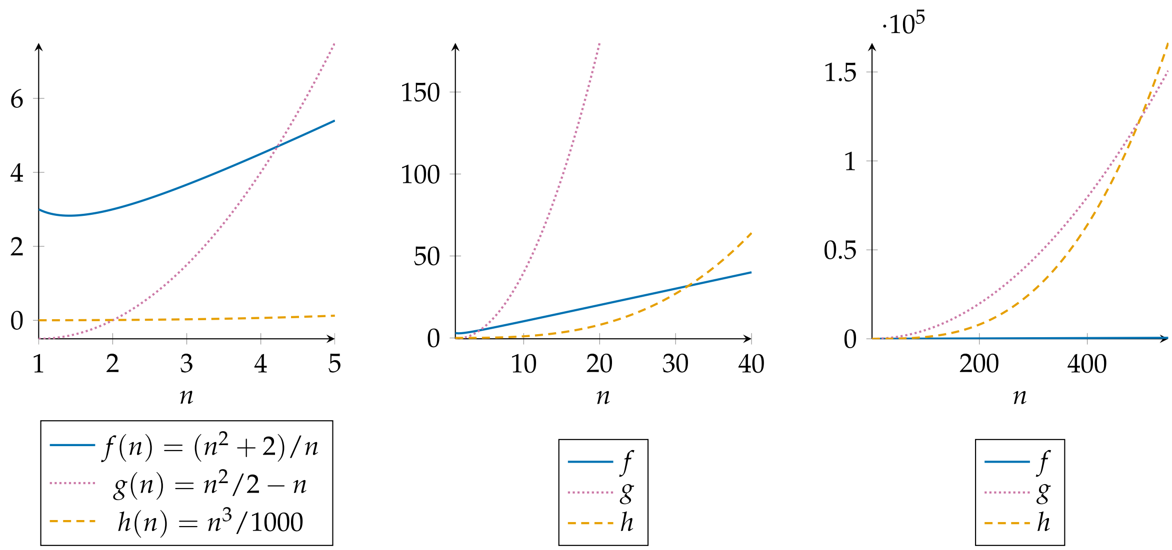
Algorithms: Asymptotic Analysis

Model 1: Big-O and Big-Ω





Critical Thinking Questions I

Learning objective: Students will describe asymptotic behavior of functions using big-O, big-Θ, and big-Ω notation.

1 Based on the Venn diagram in the model, say whether each function is $O(n^2)$, $\Omega(n^2)$, or both.

(a) $2\sqrt{n}$

(b) n^3

(c) $2n^2 + n + 1$

(d) 2^n

Consider the functions

$$f(n) = (n^2 + 2)/n,$$
$$g(n) = n^2/2 - n, \text{and}$$
$$h(n) = n^3/1000$$

for which graphs are shown in the model.

2 On each of the following intervals, list the functions f, g, and h from largest to smallest.

(a) $n \in [2, 4]$

(b) $n \in [5, 30]$

(c) $n \in [35, 450]$

3 Which function is largest, and which the smallest, at $n = 600$?

4 Does this relative order continue for all $n \geq 600$, or do the functions ever change places again? Justify your answer.

5 How do you think your answers to the previous questions relate to whether each of f, g, and h is $O(n^2)$, $\Omega(n^2)$, or both?

Say whether you think each of the following statements is true or false. Give a short justification for each answer.

6 If $f(n)$ is $O(n^2)$, then it has n^2 in its definition.

7 If $f(n)$ has n^2 in its definition, then $f(n)$ is $O(n^2)$.

8 If $f(n)$ is both $O(n^2)$ and $\Omega(n^2)$, then it has n^2 in its definition.

9 If $f(n) \leq n^2$ for all $n \geq 0$, then $f(n)$ is $O(n^2)$.

10 If $f(n)$ is $O(n^2)$, then $f(n) \leq n^2$ for all $n \geq 0$.

11 If $f(n) \leq n^2$ for all n that are sufficiently large, then $f(n)$ is $O(n^2)$.

12 If $f(n)$ is $O(n^2)$ and $g(n)$ is $\Omega(n^2)$, then $f(n) \leq g(n)$ for all $n \geq 0$.

13 Every function $f(n)$ is either $O(n^2)$ or $\Omega(n^2)$ (or both).

14 Using one or more complete English sentences and appropriate mathematical formalism, propose a definition of $O(n^2)$ by completing the following statement.

A function $f(n)$ *is* $O(n^2)$ *if and only if. . .*

Model 2: Definitions

Definition 1 (Big-O). $T(n)$ is $O(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) \leq c \cdot g(n).$$

Definition 2 (Big-Omega). $T(n)$ is $\Omega(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) \geq c \cdot g(n).$$

Definition 3 (Big-Theta). $T(n)$ is $\Theta(g(n))$ if it is both $O(g(n))$ and $\Omega(g(n))$.

Sample proof that $n^2 + 2n$ is $\Theta(n^2)$:

- First, $n^2 + 2n \leq n^2 + 2n^2 = 3n^2$ for $n \geq 1$ (since $n^2 \geq n$ when $n \geq 1$). Hence $n^2 + 2n$ is $O(n^2)$ according to the definition if we pick $c = 3$ and $n_0 = 1$.
- Next, $n^2 + 2n \geq n^2$ as long as $n \geq 0$. So by picking $c = 1$ and $n_0 = 0$, we see that $n^2 + 2n$ is also $\Omega(n^2)$.

15 Compare our class consensus definition of $O(n^2)$ with the formal definition of $O(g(n))$ above. List one way in which they are similar, and one way in which they are different.

16 Consider the following three more intuitive phrasings. Match each one with its corresponding definition.

- $T(n)$ is eventually bounded below by some constant multiple of $g(n)$.
- $T(n)$ is eventually bounded between two constant multiples of $g(n)$.
- $T(n)$ is eventually bounded above by some constant multiple of $g(n)$.

17 Which part of the definitions corresponds to the word "eventually" in Question 16?

18 In the sample proof that $n^2 + 2n$ is $O(n^2)$, the given values of c and n_0 are not the only values that would work. Given an alternate proof that $n^2 + 2n$ is $O(n^2)$ using different values of c and n_0.

19 Prove that $f(n) = 20n - 1$ is $O(n^2)$ by applying the formal definition.

20 Prove that $f(n) = n^3/10$ is $\Omega(n^2)$ by applying the formal definition.

21 Prove that $f(n) = 3n^2 - n + 1$ is $\Theta(n^2)$ by applying the formal definition.