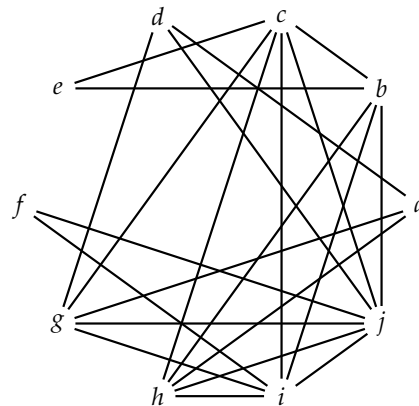
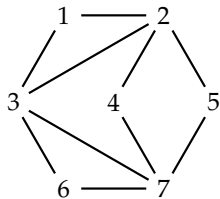


Algorithms: Introduction to reductions

Model 1: Independent sets

Definition 1. An *independent set* in an undirected graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that no two vertices in S are adjacent.

Definition 2. A *vertex cover* in an undirected graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that every edge $e \in E$ has at least one endpoint in (is “covered by”) C .



1 Which of the following are independent sets?

- (a) $\{1, 2\}$
- (b) $\{1, 5\}$
- (c) $\{c, a\}$
- (d) $\{e, a, i, g\}$
- (e) $\{7\}$
- (f) \emptyset

2 For each graph, list at least three other examples of independent sets.

3 Given an arbitrary graph G , does G always have at least one independent set? Why or why not?

- 4 Intuitively, which is harder: to find big independent sets, or small ones? Why?

- 5 Based on the previous observation, an interesting question to ask about a given graph G is to find the _____.

- 6 Try to answer your interesting question for the given example graphs (but don't spend more than a few minutes). How sure are you about your answer?

- 7 Describe a brute-force algorithm to answer this question. What is its big- Θ running time in terms of $|V|$ and $|E|$?

- 8 Guess the running time (in terms of $|V|$ and $|E|$) of the fastest known algorithm to solve this problem. (You do not have to come up with an algorithm; just guess how fast you think this problem can be solved.)

- 9 Which of the following are vertex covers?
 - (a) $\{3, 4, 5, 6, 7\}$
 - (b) $\{2, 3, 4, 6, 7\}$
 - (c) $\{b, d, e, f, g, h, i, j\}$
 - (d) $\{b, c, d, f, h, j\}$
 - (e) $\{1, 2, 3, 4, 5, 6\}$
 - (f) $\{1, 2, 3, 4, 5, 6, 7\}$

- 10 For each graph, list at least three other examples of vertex covers.



- 11 Given an arbitrary graph G , does G always have at least one vertex cover? Why or why not?
- 12 Intuitively, which is harder: to find small vertex covers, or big ones? Why?
- 13 Based on the previous observation, an interesting question to ask about a given graph G is to find the _____.
- 14 Answer your interesting question for the given example graphs. How sure are you about your answer?
- 15 Describe a brute-force algorithm to answer this question. What is its big- Θ running time in terms of $|V|$ and $|E|$?
- 16 Compare your answers to questions 1 and 9. What do you notice?

Make a conjecture based on your observations in the previous question, and prove it:

Theorem 3. Let $G = (V, E)$ be an undirected graph, and $S \subseteq V$ a subset of its vertices. Then S is an independent set if and only if _____.

Proof. (\implies) Let S be an independent set. We must show

_____. So pick an arbitrary edge $e = (u, v) \in E$;

by definition we must show that at least one of u or v _____,

that is, at least one of u or v is not _____.

Since S is an independent set and u and v are connected by an



edge, u and v can't both _____,

and therefore _____.

(\Leftarrow) (You fill in the proof for this direction!)

□

Write down what you get to assume and what you are trying to prove, and expand definitions.

