#12: Trigonometric Identities

November 28, 2008

many fascinating trig relationships

As I hinted before, there are many fascinating relationships among trigonometric functions, far too many to cover in even several assignments! (For example, see `http://en.wikipedia.org/wiki/List_of_trigonometric_identities`.) This week you'll learn just a few of the most important relationships—those that will come in handy over and over when manipulating trigonometric functions.

1 Symmetric and cofunction identities

periodic identities

There are several identities which you have already discovered in previous assignments, but are worth repeating here so they are all in one place. These identities are directly related to the definitions of sine and cosine.

First, you may recall that the cosine of a negative angle is always the same as the cosine of the corresponding positive angle:

$$\cos(-\theta) = \cos(\theta) \tag{1}$$

We call any function $f(x)$ with this property (that $f(-x) = f(x)$) an *even* function, so the above equation could be stated as "cosine is even."

You may also recall that the sine of a negative angle is the negative of the sine of the corresponding positive angle:

$$\sin(-\theta) = -\sin(\theta) \tag{2}$$

We call any function with this property *odd*, so the above equation could also be stated as "sine is odd."

Problem 1. In a previous assignment, you discovered that the graphs of cosine and sine are identical, except for the fact that the graph of sine is shifted $\pi/2$ radians to the right of cosine. Can you express this fact with an equation relating sin and cos?

2 Pythagorean identities

a Pythagorean identity

Consider Figure 1, which shows an angle of θ in standard position on a unit circle. Note, θ could be *any* angle; it may look close to $\pi/4$ radians, but you should ignore that.

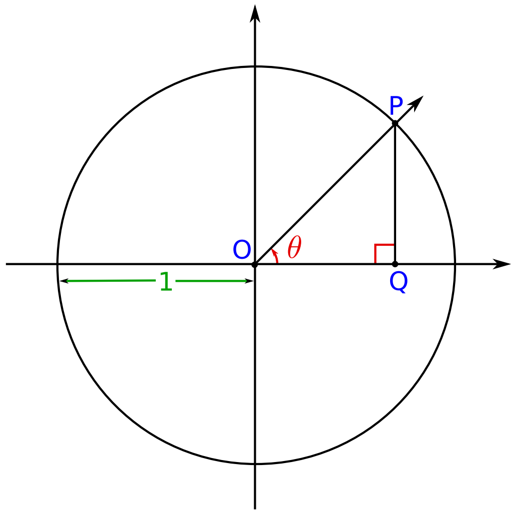

Figure 1: An angle θ in standard position on a unit circle.

Problem 2. What is the length of segment OQ in terms of θ? (*Hint*: remember your fundamental fundamentals!)

Problem 3. What is the length of segment PQ in terms of θ?

Problem 4. I won't insult your intelligence by asking you what the length of segment OP is. Since triangle OPQ is a right triangle and you now know the lengths of its three sides, what can you conclude by the Pythagorean Theorem?

Your answer to Problem 4 is known as a *Pythagorean identity* (for hopefully obvious reasons), and is an extremely fundamental trigonometric identity that comes up all over the place!

Notes on notation

You should note that $\sin^2\theta$ and $\cos^2\theta$ are commonly-used abbreviations for $(\sin\theta)^2$ and $(\cos\theta)^2$, respectively. You will often see the former written down, but you have to type the latter into your calculator (if you try typing $\sin^2 x$ into your calculator, it will get mad at you).

Problem 5. Are $\sin^2(x)$ and $\sin(x^2)$ the same? If not, what is the difference?

Problem 6. When is the sine of an angle the same as its cosine? This problem will walk you through using the Pythagorean identity to solve this equation:

$$\sin\theta = \cos\theta.$$

(a) First, square both sides, and write down the equation which results. Note that this may result in some extra "solutions" which are not actually valid solutions, so we will have to check the solutions we get at the end.

(b) Rearrange the Pythagorean identity you found in Problem 4 to solve for $\sin^2(\theta)$ in terms of $\cos^2(\theta)$, and substitute into the equation from part 1. Write down the equation which results.

(c) Simplify and solve for θ using arccosine.

(d) Which solutions are valid for the original equation?

Other Pythagorean identities

We can also use the basic Pythagorean identity you found in Problem 4 to derive a few other related identities.

Problem 7. Starting from the Pythagorean identity that you found in Problem 4, divide both sides by $\sin^2(\theta)$. Can you simplify the resulting equation? (*Hint*: think about the other trigonometric functions you learned about—tan, csc, sec, and cot.)

Problem 8. Now divide both sides by $\cos^2(\theta)$ instead. Can you simplify the resulting equation?

All three of these equations are known as Pythagorean identities, since they arise from the Pythagorean theorem, and are in some sense all equivalent. The first one involving sine and cosine is by far the most useful, but it's good to be aware of the other two as well.

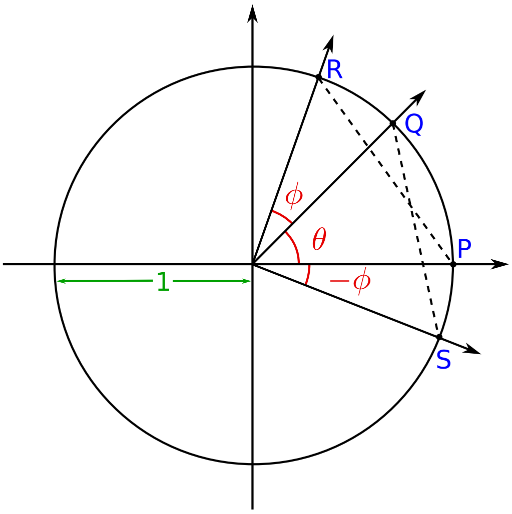

Figure 2: Proving an angle sum identity for cosine.

3 Sum and difference identities

Suppose we have two angles, θ and ϕ. What can we say about the sine or cosine of their sum or difference, $\theta \pm \phi$?

Deriving cosine sum identity

Consider the diagram in Figure 2. θ and ϕ could be any arbitrary angles.

Problem 9. First, let's figure out coordinates of points in the diagram.

(a) What are the coordinates of point P?

(b) What are the coordinates of point Q in terms of θ?

(c) What are the coordinates of point S in terms of ϕ? (Keep in mind that the angle from P to S is a *negative* angle.)

(d) What are the coordinates of point R in terms of θ and ϕ? (*Hint*: the angle from P to R is $\theta + \phi$.)

Problem 10.

(a) Use the distance formula to write down an expression for the length of segment RP (shown by a dotted line).

(b) Use the distance formula to write down an expression for the length of segment QS.

(c) Segments RP and QS are the same length, since the angle they subtend is the same. Set the expressions from parts 1 and 2 equal, solve for $\cos(\theta + \phi)$, and use the symmetric and Pythagorean identities to help you simplify!

In Problem 10, you should have come up with the fact that

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi). \tag{3}$$

Other sum and difference identities

From this, using the symmetric and cofunction identities, it's not hard to derive other laws for addition and subtraction of angles:

$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) \tag{4}$$
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \tag{5}$$
$$\sin(\theta - \phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi) \tag{6}$$

although I won't make you do so. (It's not hard, though: for example, to derive (4), substitute θ and $-\phi$ into (3), and simplify using the symmetric identities.)

Problem 11. What is the exact value of $\sin(5\pi/12)$? (*Hint*: $5/12 = 1/4 + 1/6$.)

4 Other identities

Other identities

Other identities can also be derived, such as half- and double-angle identities, power-reduction identities, and product-to-sum or sum-to-product identities. See `http://staff.jccc.net/swilson/trig/anglesumidentities.htm` for more information.

Problem 12. Visit the above website, and pick one of the identities that we haven't discussed on this assignment. Write it down and explain how it can be derived.

