#14: Polar coordinates

January 9, 2009

yo yo yo and a bottle of gum

One sunny Saturday afternoon, you are out hunting for buried treasure, as usual. It so happens that at this particular time, on this particular Saturday, you are standing immediately above the treasure (consisting of $15 trillion billion dollars' worth of vintage yo-yos and vintage bubblegum in a vintage coke bottle) but you don't know it, because your map is wrong. Or maybe you read the map wrong. Or maybe the directions you got from that nice man with the penguin were wrong. Come to think of it, his penguin *did* seem a little confused.

In any event, your map says to go 50 feet east and 20 feet north, even though ideally it should say "dig here to become filthy rich."

Problem 1. If you follow your map's instructions, how far away will you be from the treasure?

Problem 2. Being the clever lads that you are, you realize that you could save some time by walking directly to the point 50 feet east and 20 feet north of you, instead of first walking 50 feet east and then turning and walking 20 feet north. In what direction should you face before you begin walking? Give your answer as an angle in radians counterclockwise from due east. (For example, due north would be $\pi/2$.)

oops

(Unfortunately, it turns out that your map was cleverer than you, and you end up walking directly into a gorse[1]-thicket which you could have avoided by first walking 50 feet east and then turning and walking 20 feet north. Oh well.)

1 Two-dimensional coordinate systems

There are two important and commonly used systems for thinking about position in a two-dimensional world: *Cartesian* (or *rectangular*) coordinates, and *polar* coordinates.

[1] any spiny shrub of the genus *Ulex*, of the legume family, native to the Old World, especially *U. europaeus*, having rudimentary leaves and yellow flowers and growing in waste places and sandy soil

1.1 Cartesian coordinates

Cartesian coordinates

You are already familiar with the Cartesian coordinate system (also sometimes called the *rectangular* coordinate system): this is the familiar (x, y) system that you learned in fifth[2] grade: points in the plane are represented by x and y coordinates representing their *horizontal* and *vertical* distance from a distinguished point called the *origin*. Cartesian coordinates are what your map used when it told you to go to a point 50 feet east and 20 feet north of you: in other words, it told you to go to the point $(50, 20)$.

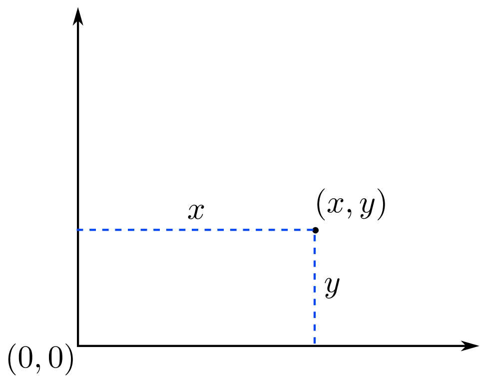

Figure 1: Cartesian coordinates

a really smart dude from the past

The Cartesian coordinate system is named for René Descartes (1596–1650), a French philosopher, mathematician, and scientist who is known as (among other things) the "father of analytic geometry." *Analytic geometry* simply refers to the idea of using a numerical coordinate system to analyze geometrical; this may seem obvious to you now (having been taught it in school), but in its time it was a brilliant new idea that directly led to the development of calculus by Newton and Leibniz.

Problem 3. Can there be two pairs of (x, y) coordinates that refer to the same point? Why or why not?

1.2 Polar coordinates

polar coordinates

The second coordinate system is the *polar* coordinate system. In this system, points in the plane are represented by their *distance* from the origin, and the *angle* that they make with the positive x-axis. (Actually, calling it the

[2] something like that

"positive x-axis" is kind of silly, since x specifically has to do with Cartesian coordinates; in the polar coordinate system, we call it the *pole*.) Polar coordinates are what you used when you computed a shortcut through the gorse-thicket: you figured out the *distance* and *angle* at which you had to travel to get to the desired point. In a polar coordinate system, the distance from the origin to a point is called r (R for Radius) and the angle from the pole is called θ.

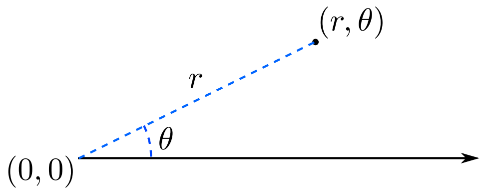

Figure 2: Polar coordinates

Problem 4. In which quadrant is the point with polar coordinates $(10, 4\pi/3)$?

Problem 5. Can the same point have multiple different polar coordinate pairs? If not, explain why; if yes, give an example.

2 Conversion

converting between Cartesian and polar

Since we have two different ways to refer to points in the plane, it's useful to know how to convert between the two representations. Everything you need to know about the relationship between Cartesian and polar coordinates is shown in the diagram in Figure 3.

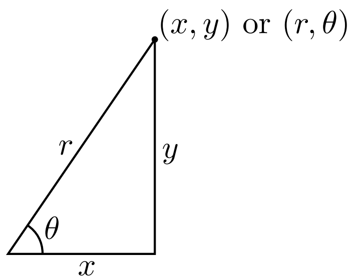

Figure 3: The relationship between Cartesian and polar coordinates

The point at the upper right corner of the triangle has Cartesian coordinates (x, y) and polar coordinates (r, θ).

Problem 6. Use Figure 3, your knowledge of trigonometry, and the Pythagorean Theorem to help answer the following questions:

(a) What is x in terms of r and θ?

(b) What is y in terms of r and θ?

(c) What is r in terms of x and y?

(d) What is θ in terms of x and y?

Problem 7. Convert from polar to Cartesian coordinates.

(a) $(\sqrt{2}, \pi/4)$

(b) $(5, 4\pi/3)$

(c) $(2, \pi)$

(d) $(6, 0)$

Problem 8. Convert $(-5, \pi/2)$ from polar to Cartesian coordinates. Does this make sense? What should negative values of r mean?

Problem 9. Convert from Cartesian to polar coordinates.

(a) $(-1, -1)$

(b) $(0, 2)$

(c) $(50, 20)$

3 Polar graphs

Cartesian vs polar graphs

As you know, we can graph any equation involving x and y by drawing a line through all the points whose x and y coordinates satisfy the equation. For example, the graph of $y = 2x + 1$ is a line; the graph of $y = \cos x$ is a squiggly wave; and the graph of $x^2 + y^2 = 16$ is a circle with radius 4. We can do the same thing with equations involving r and θ.

Problem 10. What do you think the graph of $r = 4$ looks like? (Which points have polar coordinates with $r = 4$?)

Problem 11. What do you think the graph of $r = \theta$ looks like?

use the graphing calculator, Luke

Your graphing calculator can probably make polar graphs! Go to the "mode" screen and look for something like "Polar" or "Pol" mode. Now when you go to the screen where you can enter equations to be graphed, it should say "$r =$" instead of "$y =$"! You can now type equations in terms of θ (which you can type with the same button you would otherwise use for x).

Problem 12. Try graphing each of the following polar equations using your graphing calculator, and describe what they look like.

(a) $r = \cos\theta$

(b) $r = \sin\theta$

(c) $r = \cos(5\theta)$

(d) $r = 1 + \cos\theta$

(e) $r = 1 + 2\cos\theta$

Problem 13. Try graphing some polar equations of your own, and write down one of your favorites.

