#16: Complex Numbers (review)

January 23, 2009

For the next couple weeks we will study the *complex* numbers, $\mathbb{C}$. You've probably seen complex numbers before, so much of this week's assignment might be review, but hopefully you'll learn some new things as well. Next week, however, you'll learn some amazing things about complex numbers which I guarantee you don't already know: complex numbers, like vectors, can be represented both in rectangular and polar form, and the polar form is totally sweet.[1]

1 Imagine this

Suppose someone asked you to solve the equation

$$x^2 - 4 = 0.$$

"That's easy," you respond, "$x = \pm 2$."

"Okay," your mysterious interlocutor continues, "how about

$$x^2 - 2 = 0?"$$

"No problem," you say, "

Problem 1.

"

The unknown interrogator presses, "Aha, but what about

$$x^2 + 1 = 0?"$$

"Erm..." you stammer. "Everyone knows that equation has *no solutions*." You see, this is taking place 400 years ago.[2]

[1] Oops, I shouldn't have said anything. Now you probably won't be able to sleep all week.

[2] Didn't you know that?

"But isn't that weird?" the aetherial inquisitor inquires. "Some polynomial equations have solutions, and some don't! It seems so wrong… so… *inelegant*."

You concede that he has a good point, mumble some sort of excuse involving a dentist, and leave as quickly as possible.

history of imaginary numbers

Funny as this sounds, it was the basic state of affairs in mathematics until the 1700s or so. Gerolamo Cardano invented imaginary numbers in the mid-1500s, but he didn't really understand them, and only used them as tools for solving equations which actually had real solutions. They didn't become widely accepted until Euler and Gauss used them in the mid-1700s.

The interesting thing is where imaginary numbers got the name "imaginary." René Descartes[3] was the first person to use that name for them, in 1637—but he was using that name to *make fun of them!* Like many other mathematicians of the time, he dismissed them as stupid, useless, and, well… *imaginary*.

imaginary numbers aren't imaginary

But "imaginary" numbers are actually no more imaginary than, say, negative numbers. You cannot actually have *negative three rocks*. In some sense, "negative three" is an abstract, imaginary concept that doesn't correspond to anything in the "real world." But in another sense, it does correspond to things in the real world—for example, if you owe someone three dollars, the situation can be correctly and usefully modelled by the number "negative three."

well, ok, they are, but so is everything else

Fine, you say, but surely *imaginary* numbers don't usefully model anything in the real world. Ah, but they do! For example, the voltage of the alternating current that is at this very moment providing power to your computer can be represented by a complex number. The quantum interactions of subatomic particles can also be modeled using complex numbers. And there are probably many other such things that I don't know about. The point is, once you get much beyond numbers like 1, 2, 3, … everything (fractions, negative numbers, real numbers, humongously large integers…) is pretty much imaginary. But that doesn't make them useless or uninteresting.

the imaginary unit i

Anyway, the key point to finding solutions for all those "unsolvable" equations, of course, is to introduce a new number called i. This *imaginary unit* has the property that

$$i^2 = -1. \tag{1}$$

[3] the same dude you learned about in assignment #14

Problem 2. Explain why there is no *real number* which is a solution to equation (1).

In general:

imaginary numbers

> An *imaginary number* is any number of the form ai, where a is a real number, and i is an *imaginary unit* satisfying equation (1).

It turns out that throwing i in alongside the real numbers means that now *all* polynomial equations have solutions!

Problem 3. Prove that the square of every nonzero imaginary number is a negative real number.

Problem 4.

(a) You know what i^2 is. What is i^3?

(b) What about i^4?

(c) What is i^{97}?

Problem 5. Simplify: $\sqrt{-12}$.

2 Complex numbers

If 3 is a number, and i is a number, what happens when we add them? We can fix everything up by generalizing one more time, to *complex numbers*. A complex number has a real part and an imaginary part.

complex numbers

> A *complex number* is the sum of a real number and an imaginary number; in other words, something of the form $a + bi$, where a and b are real numbers. The set of all complex numbers is denoted $\mathbb{C}$ (LaTeX: `\C`).

Again, "complex" is probably a bad name; there's nothing particularly complex about the complex numbers. They're actually pretty simple.

Keep in mind that the b in $a + bi$ can be negative. For example, $3 - 4i$ is a complex number; it is in the form $a + bi$ where $a = 3$ and $b = -4$.

Problem 6. Is 3 a complex number? (*Hint:* this is a trick question, think carefully.)

Problem 7. Solve for x:

(a) $x^2 - 4x + 5 = 0$

(b) $x^2 + x + 1 = 0$

3 Complex arithmetic

basic rules for complex numbers

Of course, it's no use having complex numbers if we can't do things with them. Thankfully, manipulating complex numbers is very easy, if you just keep in mind the following two rules:

1. $i^2 = -1$.
2. Combine terms with i (imaginary parts) and terms without i (real parts) separately.

complex addition

So, to add two complex numbers, just add the real parts, and add the imaginary parts. For example, $(2 + 3i) + (4 - 6i) = 6 - 3i$.

Problem 8. Simplify:

(a) $(19 - i) + (-2 + 6i)$

(b) $(i - 4) + (5 - 2i)$

(c) $(3 + i) - (4 + 6i)$

(d) $(-5 + 2i) + 3(2 - i) - (4 - 4i)$

complex multiplication

To multiply complex numbers, just expand out the multiplication, remember that $i^2 = -1$, and collect like terms. For example:

$$\begin{aligned}(2+3i)(4-i) &= (2)(4) + (2)(-i) + (3i)(4) + (3i)(-i) \\ &= 8 - 2i + 12i - 3i^2 \\ &= 8 - 2i + 12i + 3 \\ &= 11 + 10i\end{aligned}$$

Problem 9. Simplify:

(a) $(2+i)(3+2i)$

(b) $(2+i)^3$

(c) $(6+2i)(6-2i)$

Dividing complex numbers is a bit trickier; to see how to do it, we'll first need to explore the concept of *complex conjugates*.

4 Complex conjugates

conjugate

> The *conjugate* of the complex number $a + bi$ is $a - bi$.

Of course, b could be negative, so we can also say that the complex conjugate of $a - bi$ is $a + bi$. The point is that to find the conjugate, you switch the sign of the imaginary part.

Problem 10. What is the complex conjugate of 3?

Problem 11. What is the complex conjugate of $i + 4$?

Problem 12. Prove that the product of any complex number with its conjugate is a real number.

5 Complex division

complex division

Now we are in a position to talk about division of complex numbers. The idea is that we start with a fraction of the form

$$\frac{a+bi}{c+di},$$

and to simplify this we simply get rid of any imaginary parts in the denominator. Of course, in Problem 12 you discovered a good way to do this: just multiply a complex number by its conjugate and it becomes a real number, with no imaginary parts. We can't just multiply the denominator by its conjugate—but if we multiply the numerator *and* denominator by the conjugate of the denominator, then we're just multiplying by one and everything is OK! Here's an example:

$$\begin{aligned}\frac{2+3i}{2-2i} &= \frac{2+3i}{2-2i}\cdot\frac{2+2i}{2+2i} \\ &= \frac{4+6i+4i-6}{4-4i+4i+4} \\ &= \frac{-2+10i}{8} \\ &= \frac{-1+5i}{4}\end{aligned}$$

And now we're done, because we've simplified the division to a single complex number (with $a=-1/4$ and $b=5/4$).

Problem 13. Simplify:

(a) $(-1+5i)/(1+i)$

(b) $(3-4i)/(2-i)$

(c) $5/i$

