

#17: The Polar Form of Complex Numbers

January 30, 2009

Now that you have a foundation in the basics of complex numbers, this week, as promised, we're going to explore the extremely fascinating *polar representation* of complex numbers.

But first, a digression...

1 The number line

"The number line?" I hear you exclaim. "That's, like, SO first grade. Why are we learning about the number line?" Well, bear with me for a minute...

the number line

As you no doubt learned in first grade (with successive refinements to include negative numbers, and then fractions, and then real numbers), we can think of the real numbers as inhabiting a one-dimensional space called the *number line*. Figure 1 illustrates the set of integers \mathbb{Z} located at discrete points on a number line; the real numbers, of course, inhabit the entirety of the line.

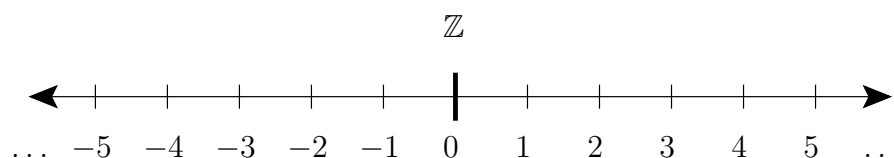


Figure 1: The number line

arithmetic as geometry

So, why is this useful or interesting? The key point is that *simple arithmetic operations on the real numbers have simple geometric interpretations on the number line*.

addition is translation

For example, as illustrated in Figure 2, addition corresponds to *translation*. If you start with some number n , adding four to it corresponds to moving four units right along the number line. Likewise, subtracting four (that is, adding negative four) corresponds to moving four units left.

multiplication is scaling/reflection

Multiplication by a positive number corresponds to *scaling*; multiplication by a negative number also corresponds to *reflection*.

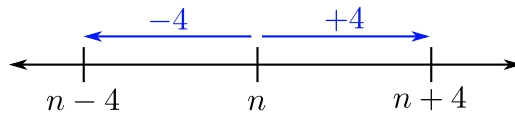


Figure 2: Real addition corresponds to translation on the number line

Problem 1. Draw pictures (something like Figure 2) that illustrate the scaling and reflection properties of multiplication on the number line. You can submit these pictures in one of two ways:

1. If you draw them using some sort of computer program, you can email them along with your assignment.
2. If you draw them on paper, you can mail them to me. (In this case, just mail them by the due date for your assignment; it doesn't matter when they reach me.)

What about complex numbers? Can we come up with a spatial model for the complex numbers, where complex arithmetic has a nice geometric interpretation?

2 The complex plane

the complex plane

Just as we can picture the integers or real numbers as being located on a one-dimensional structure, the *number line* or *real number line*, we can picture the complex numbers in the two-dimensional *complex plane*, as shown in Figure 3.

real and imaginary axes

The real numbers are located along the horizontal axis (marked **R** in the picture above¹), and the imaginary numbers are located along the vertical axis (marked **I**). So instead of the *x*-axis or *y*-axis we talk about the *real axis* and the *imaginary axis*. Zero, which is of course both a real number and an imaginary number, is at the origin, the intersection of the real and imaginary axes. So, for example, the complex number $3 + 2i$ is located three units right and two units up from the origin, exactly where $(3, 2)$ would be located if we were talking about the Cartesian plane. (In some sense, we *are* talking about the Cartesian plane.)

complex arithmetic as geometry?

So, this *seems* rather natural, but it would be really nice if complex arith-

¹It really should be marked \mathbb{R} instead of **R**, but... well, it's a long story.

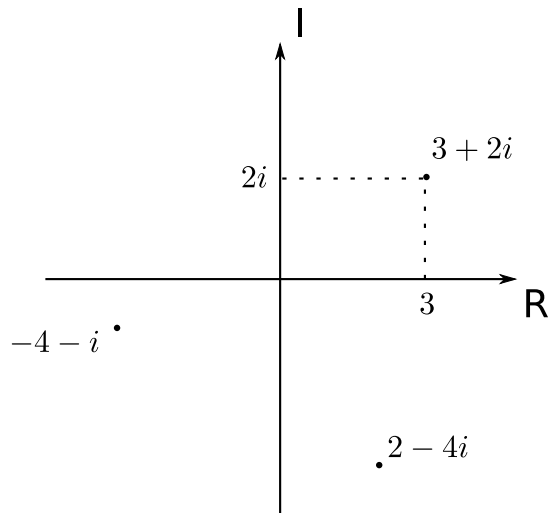


Figure 3: The complex plane

metric operations had geometric interpretations in the complex plane, just as real arithmetic has geometric interpretations on the number line.

Well, for addition, it turns out that this isn't too hard:

Problem 2. What is a geometric interpretation of complex addition? In other words, if you start at some point in the complex plane and add $a + bi$, where do you end up?

However, multiplication is not as obvious!

Problem 3. Get out a piece of graph paper and plot each of the following points in the complex plane (you may want to use a different graph for each subproblem, so you can see what's going on).

- (a) i, i^2, i^3, i^4
- (b) $3 + 2i, -2 + 1$, and $(3 + 2i)(-2 + i)$
- (c) $4i, -1/2, (-1/2) \cdot 4i$

Problem 4. Before you read on, write down some of your observations or guesses about what multiplication does geometrically in the complex plane.

To really see what is going on with multiplication, we'll have to talk about...

3 Complex numbers in polar form

*complex numbers
using polar
coordinates*

Now that we are thinking of complex numbers as occupying points on a plane, an idea naturally suggests itself: what would happen if we thought of complex numbers in terms of polar coordinates, instead of Cartesian coordinates?

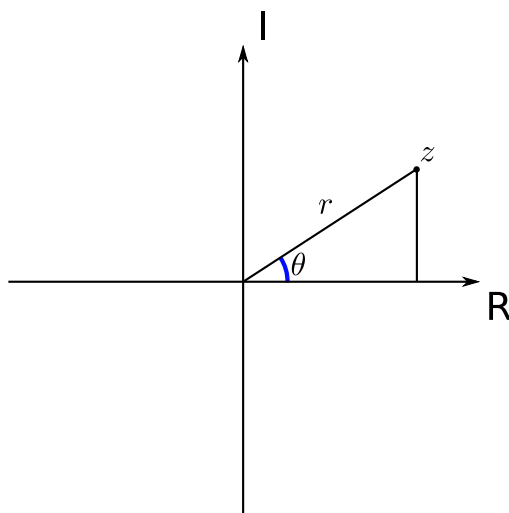


Figure 4: Representing complex numbers in polar coordinates

Problem 5. Consider Figure 4. It shows some complex number z in the complex plane, with its distance r and angle θ from the origin labeled. Write z in Cartesian $(a + bi)$ form, in terms of r and θ .

*a Really Amazing
Fact*

And now for a Really Amazing Fact: it turns out that

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (1)$$

huh?

Now, you might very well ask whether I am pulling your leg. What does it even *mean* to raise e to the power of an *imaginary* number!? Unfortunately, there isn't room on this assignment to explain—and even if there were, every way I know how to explain it requires calculus! But you can take my word for it that if you think about what raising e to an imaginary power *could possibly* mean, there is only one possible definition that works out correctly with everything else we already know about e , exponentiation, imaginary numbers, sine, cosine, and so on—and that definition is equation (1)! Hopefully you can see how surprising and beautiful this equation is. Why should

cosine and sine show up if you raise e to an imaginary power? Before seeing this equation, you would have no reason to suspect that e , cosine, and sine even have anything at all to do with one another.

a beautiful equation

Problem 6. Substitute $\theta = \pi$ into equation (1), and simplify using your knowledge of sine and cosine. Now add one to both sides, and write down the equation you get. This is called *Euler's identity*. Why do you think this is considered one of the most beautiful equations in all mathematics?

the polar form of a complex number

Problem 7. Use equation (1) to rewrite your answer to Problem 5 in terms of r , θ , i , and e .

Your answer to Problem 7 is the *polar form* of a complex number. Although it's difficult to add two complex numbers in polar form (it would be easier to just convert them to Cartesian form first), the polar form makes multiplication very easy!

multiplication of polar forms

Problem 8. Using the polar form you found in Problem 7, show that if one complex number is r_1 units away from the origin at angle θ_1 , and another complex number is r_2 units away from the origin at angle θ_2 , then their product is $r_1 r_2$ units away from the origin at an angle $\theta_1 + \theta_2$.

complex multiplication is rotation!

This, then, is the answer: in the complex plane, multiplication corresponds not just to *scaling* (like it did on the number line) but also to *rotation*! For example, multiplying by i corresponds to a counterclockwise rotation of $\pi/2$ (90°) in the complex plane. And multiplying by $2e^{i\pi/4} = \sqrt{2} + i\sqrt{2}$ corresponds to a rotation by $\pi/4$ and scaling by 2—that is, any complex number, when multiplied by $\sqrt{2} + i\sqrt{2}$, will end up rotated by an angle of $\pi/4$, and twice as far away from the origin.

Problem 9. Plot $1 + i$, $(1 + i)(\sqrt{2} + i\sqrt{2})$, $(1 + i)(\sqrt{2} + i\sqrt{2})^2$, and $(1 + i)(\sqrt{2} + i\sqrt{2})^3$ on your graph paper. What happens?

4 Beyond 2D?

quaternions!

You might well ask whether there are any sorts of numbers which can be envisioned as inhabiting some sort of space with a dimension higher than two. And the answer is... yes! There are four-dimensional numbers called *quaternions*. The quaternions still have an imaginary number i with $i^2 = -1$; but they have two additional imaginary numbers, called j and k . j^2 and k^2

are both equal to -1 , just like i^2 ; additionally, $ijk = -1$. Just as complex numbers can be written in the form $a + bi$, where a and b are real numbers, quaternions can be written in the form $a + bi + cj + dk$, where a , b , c , and d are all real numbers.

Interestingly, just as we lose a nice property when generalizing from the real numbers to the complex numbers (the fact that the real numbers are in a certain *order*, so we can talk about real numbers being less than or greater than other real numbers), we lose another property when generalizing to the quaternions: quaternion multiplication is not *commutative*, that is, if x and y are quaternions, it is not necessarily true that $xy = yx$ (this is true for complex numbers). Quaternions can represent rotations in three dimensions, so they are sometimes used in computer graphics, as well as in other applications.

There are also eight-dimensional *octonions* (octonion multiplication is not even *associative*), and sixteen-dimensional *sedonions*, but by that point so many nice properties have been lost that it's not even clear whether these things should be called "numbers" anymore!