#19: Linear Transformations

February 22, 2009

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matrices as functions

Last week, you saw that we can think of 2×2 matrices as *functions* of the Cartesian plane—that is, matrix multiplication represents some sort of two-dimensional transformation. The obvious question is: what *sorts* of two-dimensional transformations can we represent with matrices in this way?

Another important question is, given some matrix, how can we tell what transformation it represents? Last week, you looked at a few examples—but how do we know we can tell what transformation a matrix represents just by looking at its action on a few points? For example, one of the matrices you looked transformed four points making up the vertices of a square into a slightly rotated square—but how do we know this matrix would have the same effect on other points? Perhaps it only rotates points near the origin, and leaves some farther-away points alone, or maybe it rotates some points clockwise and some points counterclockwise, or maybe. . . there are endless possibilities.

matrix functions are simple

It turns out that we *can* actually tell what transformation a matrix represents by looking at only a few examples, and that there are really only a few fundamental sorts of transformations that can be represented by matrices (fortunately, they're very useful ones!). The fundamental result we will show is this:

> *If we know what a matrix does to the special points* $(1, 0)$ *and* $(0, 1)$*, then we know everything there is to know about it!*

Intuitively, this says that transformations represented by matrices can't do anything very "strange". They are so orderly and regular that we can know everything there is to know about them by just looking at what they do to two specific points. Let's see why this is true!

1 Linear transformations

scalar multiplication

Suppose s is a real number. If B is a matrix, then sB represents the matrix

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you get when you multiply each of its entries by s. For example,

$$3\begin{pmatrix} 2 & 5 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 15 \\ 21 & -3 \end{pmatrix}.$$

This is known as *scalar multiplication.* (A "scalar" is another name for the familiar sorts of numbers you are used to, like 3, π, $\sqrt{6}$—scalars are "zero-dimensional" numbers, vectors are one-dimensional, and matrices are two-dimensional. (Numbers with dimensions higher than that are called "tensors".))

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix and that $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is a 2×1 matrix representing the point (x, y).

Problem 1. What is the matrix product AX? Write your answer as a matrix in terms of a, b, c, d, x, and y.

Problem 2. Write sX as a matrix in terms of s, x, and y.

Problem 3. Compute the matrix product $A(sX)$. (Write your answer as a matrix in terms of . . . yada yada.)

Problem 4. Compute the product $s(AX)$. Show that it is equal to your answer to Problem 3.

You have just shown that for any scalar s and matrices A and X,

$$A(sX) = s(AX). \tag{1}$$

(Actually, you only showed this when A is a 2×2 matrix and X is 2×1—but it is not hard to generalize the proof to cover any matrices which can be multiplied.)

Let's prove one more property of matrices. In addition to A and X, suppose we also now have the matrix $Y = \begin{pmatrix} w \\ z \end{pmatrix}$.

Problem 5. Compute $A(X + Y)$.

Problem 6. Compute $AX + AY$, and show that it is equal to your answer from the previous problem.

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This is a nice result—matrix multiplication distributes over matrix addition, just like with normal (scalar) arithmetic!

$$A(X + Y) = AX + AY \tag{2}$$

linear transformations

The above two properties ((1) and (2)) together have a special name: we say that multiplying by a matrix is a *linear transformation.* Why is it so special?

2 Unit vectors

unit vectors and LaTeX

The vectors $(1, 0)$ and $(0, 1)$ are special—so special, that we usually give them special names, $\vec{\imath}$ and $\vec{\jmath}$. (You can write them in LaTeX as `\vec{\imath}` and `\vec{\jmath}`. `\vec` adds the little arrow on top; note that you should use `\imath` and `\jmath` inside the `\vec` instead of just `i` and `j`. Can you spot the difference between $\vec{i}$ (`\vec{i}`, wrong) and $\vec{\imath}$ (`\vec{\i}`, right)? Can you guess what `\imath` and `\jmath` do?)

$\vec{\imath}$ and $\vec{\jmath}$ are called "unit" vectors since they are one unit long. There are two of them since we are dealing with two dimensions (in three dimensions, there are three unit vectors: $\vec{\imath} = (1, 0, 0)$, $\vec{\jmath} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$).

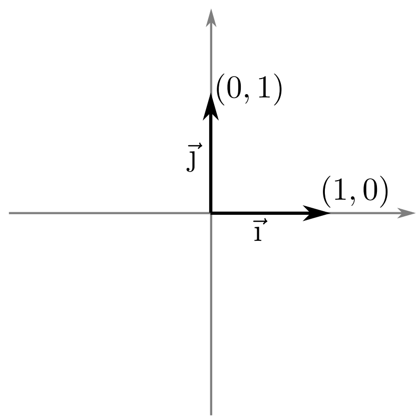

Figure 1: The unit vectors $\vec{\imath}$ and $\vec{\jmath}$

decomposing vectors with unit vectors

$\vec{\imath}$ and $\vec{\jmath}$ are special because any other vector can be written in terms of them. (Actually, it turns out that this *isn't* particularly special about $\vec{\imath}$ and $\vec{\jmath}$; you can do this with almost any two vectors, as long as they don't point in the

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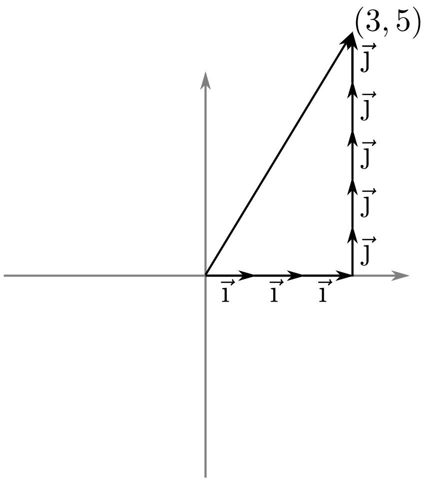

Figure 2: Expressing the vector $(3, 5)$ in terms of $\vec{\imath}$ and $\vec{\jmath}$

same direction. But at least with $\vec{\imath}$ and $\vec{\jmath}$ it's easy to see why.) For example, the vector $(3, 5)$ can be written as $3\vec{\imath} + 5\vec{\jmath}$, as shown in Figure 2.

In matrix notation, we could also write this as

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Do you see why this is true?

In general, of course,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\vec{\imath} + y\vec{\jmath}. \tag{3}$$

3 Matrix transformations, revealed

But now we are almost done!

Problem 7. Given the matrix A from section 1, what is $A\vec{\imath}$? What is $A\vec{\jmath}$?

Problem 8. Show that if $X = \begin{pmatrix} x \\ y \end{pmatrix}$, then $AX = x(A\vec{\imath}) + y(A\vec{\jmath})$. (*Hint*: use equation (3) to decompose X in terms of $\vec{\imath}$ and $\vec{\jmath}$; then use equation (1) and equation (2) to simplify.)

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matrix transformations are determined by unit vectors

But this means that no matter what X is, the transformation that A performs on X can be expressed in terms of what A does to the unit vectors $\vec{\imath}$ and $\vec{\jmath}$—so if we know $A\vec{\imath}$ and $A\vec{\jmath}$, then we know what A does to *any* point!

4 Basic linear transformations

Meet Mr. Elephant, a happy critter who lives at the origin of a Cartesian plane.

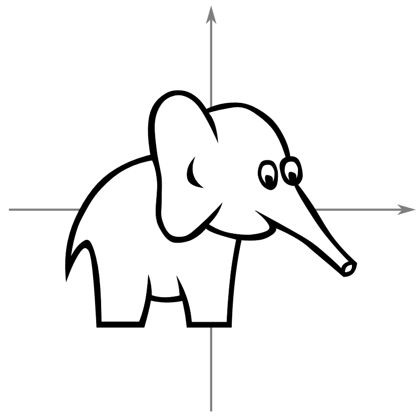

Figure 3: Mr. Elephant.

Sometimes for fun, Mr. Elephant likes to transform himself with matrices. (By "transform himself with matrices," I simply mean "multiply every point in his body by a matrix.") Can you help him?

Problem 9. Write down a 2×2 matrix which would transform Mr. Elephant into Tiny Mr. Elephant, shown in Figure 4. Tiny Mr. Elephant is exactly 1/3 the size of regular Mr. Elephant. (*Hint:* think about what such a matrix would have to do to $\vec{\imath}$ and $\vec{\jmath}$.) This type of transformation is called a *scale*.

Problem 10. Write down a 2×2 matrix which would transform Mr. Elephant into each of the forms listed below. Think about what each transformation would do to the unit vectors $\vec{\imath}$ and $\vec{\jmath}$.

(a) Fat Mr. Elephant, who is twice as wide as Mr. Elephant but the same height (Figure 5). This is also a *scale*.

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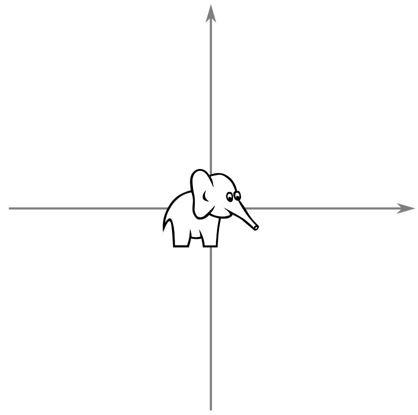

Figure 4: Tiny Mr. Elephant.

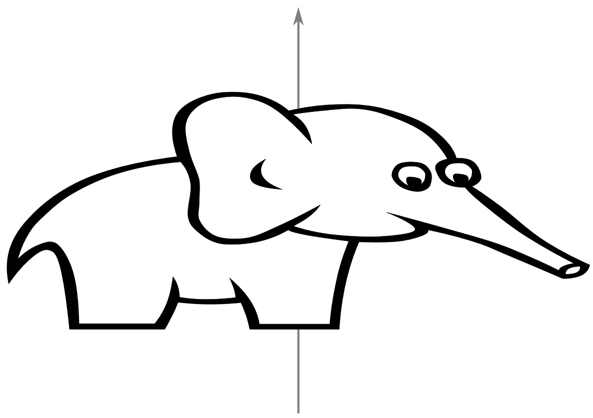

Figure 5: Fat Mr. Elephant.

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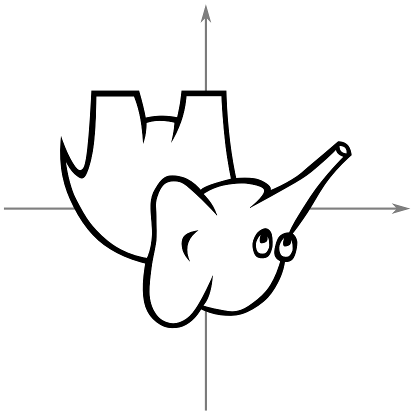

Figure 6: Upside-down Mr. Elephant.

(b) Upside-down Mr. Elephant (Figure 6). this transformation is a *reflection*—although you can also think of it as a kind of scale.

(c) Amazing Mr. Elephant, who is so amazing that he can balance on his tail (Figure 7). This transformation, of course, is a *rotation*.

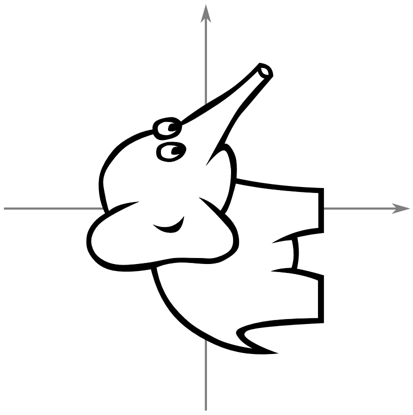

Figure 7: Amazing Mr. Elephant.

(d) Lazy Mr. Elephant (Figure 8), who is rotated 120° in order to lie on his back.

(e) Frightened Mr. Elephant (Figure 9). This sort of transformation is called a *skew* or *shear*.

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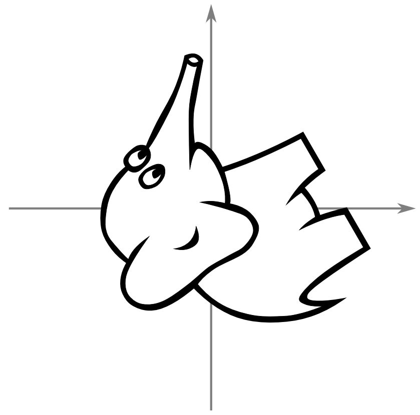

Figure 8: Lazy Mr. Elephant.

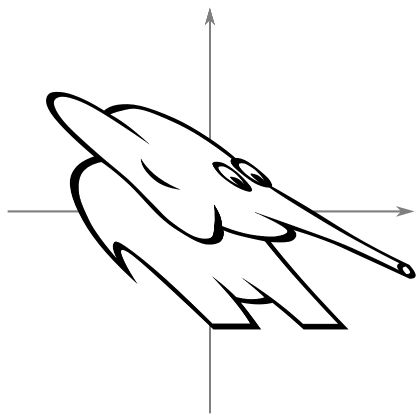

Figure 9: Frightened Mr. Elephant!

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(f) Crazy Mr. Elephant (Figure 10). (*Hint:* this is the hardest one. It might help to think about how you could make Crazy Mr. Elephant by combining some of the other transformations.)

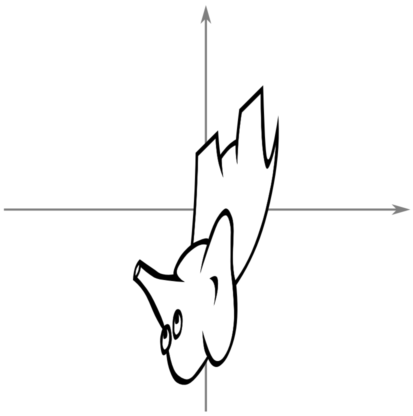

Figure 10: Crazy Mr. Elephant.

Problem 11. Is it possible to make a matrix which transforms Mr. Elephant as shown in Figure 11? If so, write down such a matrix. If not, explain why. (*Hint:* think about what a linear transformation can do to the point $(0, 0)$.)

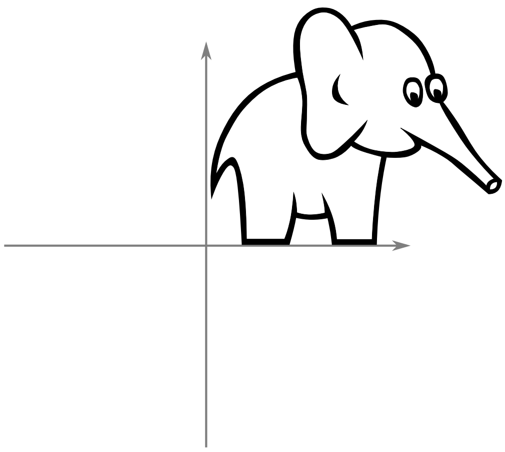

Figure 11: Translated Mr. Elephant.

Next week, the exciting conclusion—we'll see how to represent translations with matrices using so-called *affine transformations*, and then use these

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affine transformations to make ferns, sierpinski triangles, and other fantastic shapes!

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