

#20: The Art of Counting, Part I

March 7, 2009

Up until now, we have been exploring *continuous* mathematics, dealing with real numbers and measurement. This week we will begin the second half of the course, focusing on *discrete* mathematics, dealing with pattern and structure.

This week, we begin with counting. What's that? You already know how to count? Aha, that's what *you* think! Counting is about much more than just saying "one, two, three, ..." For example, suppose someone showed you a rectangular grid of dots and asked you how many dots there were. Would you just point to each and every dot and literally count them? Or would you cleverly realize that it was a *rectangular* grid, count how many dots were along each edge, and multiply to get the total number of dots? The mathematical study of counting—called *combinatorics*—takes this idea of counting things in clever ways ("counting without actually counting") to whole new levels.

1 Basic counting techniques

1.1 Alternatives = addition

For the most part, this is intuitively obvious, but it's worth stating: if the things you are trying to count can be separated into disjoint groups (*disjoint* means that the groups do not overlap), and you know how to count each group, then the total number of things is just the sum of the number of things in each group.

Problem 1. A petting zoo has sixteen rabbits, fourteen snails, and thirty-nine lemurs. How many animals are in the petting zoo?

Problem 2. How many animals are in the petting zoo now?

1.2 Independent choices = multiplication

Often when counting things it is useful to think about how you could *choose* one of the things you are counting. If choosing one of them involves making

two *independent* choices, then the total number of things is the number of ways of making the first choice, *times* the number of ways of making the second choice.

For example, if you are counting the number of dots in a rectangular grid, you could choose any particular dot by first choosing a row, and then choosing a column. Since these choices are independent (you could choose any column no matter which row you pick, and vice versa), the total number of ways to choose a dot is the number of ways to choose a row (that is, the number of rows) times the number of ways to choose a column (that is, the number of columns).

Problem 3. How many seconds are there in a day?

Problem 4. A company is trying to decide on a logo for a new product. They have narrowed things down to four different colors, three different designs, and six different names for the product. Assuming these choices are independent, how many possible logos are there?

Problem 5. How many red face cards are in a standard deck of cards?

You have to be careful, though: if the choices are not independent, you can't just multiply! You have to be a bit more clever.

Problem 6. At the Tastie-Freeze Ice Cream Store, you can choose any one of their thirty delicious ice cream flavors, and, optionally, any one of their four delicious toppings, *unless* you have vanilla, in which case you may have two toppings, or if you have Super-Chunky Peanut Butter Cup Fudge Raisin Mint Coffee Cookie Surprise, in which case you don't get a topping. How many different things can you order?

2 Permutations

Permutation is just a fancy mathematical name for *order*: to make a permutation of some objects, you just put them in some particular order. For example, if we have the letters 'A', 'B', and 'C', one permutation of them is "ACB"; another permutation is "BAC", another is "ABC", and so on.

Problem 7. How many permutations of 'A', 'B' and 'C' are there?

Permutations come up in lots of places, so it's nice to know how to count them. Suppose we have n objects, and we want to know how many permutations of the n objects there are. It's useful to again think in terms of *choosing* a permutation: how many ways can we choose a particular permutation?

Well, the permutation has to start with one of the n objects, right? So we have n choices for the first object. After that, there are $n - 1$ objects left, and we are free to choose any of them to be the second object in the permutation. Since this choice is independent of the first, there are a total of $n \cdot (n - 1)$ ways to choose the first two objects. Of course, that leaves $n - 2$ choices for the third object, and so on. . . until we are down to the last object and there is only one thing left to choose. So, there are

$$n \cdot (n - 1) \cdot (n - 2) \cdots 2 \cdot 1 = n!$$

permutations of n objects.

Problem 8. A group of seven tourists is having their picture taken in front of the Scottsbluff, Nebraska Municipal Water Tower. They are unsure what the optimal order is for them to stand in (shortest to tallest? tallest to shortest? alternating?) so they decide to take one picture with them standing in every possible order. Assuming they can take one picture every thirty seconds, how long will this take?

Problem 9. After looking at the pictures, the tourists realize that only four of them fit in the picture frame at once, so they go back to take more pictures. This time, they want to take a picture of every possible group of four of them in every possible order. How many pictures will they take?

3 Practice problems

Problem 10. The country of West Mathistan issues license plates with four letters and three numbers (for example, my car's license plate is "NERD-123"). Every car must have a different license plate. What is the maximum number of cars that can be registered in West Mathistan?

Problem 11. You have two normal six-sided dice, except that one is red and one is blue, and you roll them both.

- (a) How many different outcomes are there?
- (b) In how many different ways can you roll a total of 7?

(c) In how many different ways can you roll a total larger than 7?

Problem 12. How many seconds are in a century? (You may ignore leap seconds, and assume that exactly 24 out of 100 years are leap years.)

Problem 13. How many seconds were there between 12:00 AM on January 1, 1970, and 11:31:30 PM on February 13, 2009?

Problem 14. How many squares (of *any* size!) are there on a chess board?

Problem 15. Figure 1 below illustrates the first four *triangular numbers*; the n th triangular number is the number of dots in a triangle of size n . As you can see by counting the dots below, the first four triangular numbers are 1, 3, 6, 10. What is the 100th triangular number?

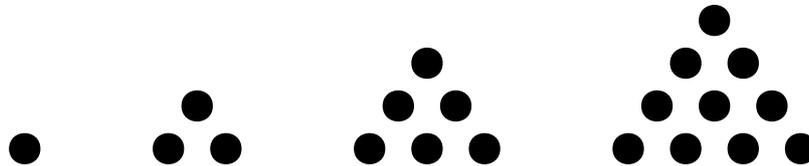


Figure 1: Triangular numbers

Problem 16. If Ronnie has six pairs of basketball shoes, nineteen pairs of special lucky basketball socks, three pairs of basketball shorts, and five basketball jerseys, how many different basketball outfits can he wear? Give as many possible answers as you can think of, by interpreting the question in different ways. (For example, is Ronnie allowed to mix and match his shoes? What about his socks? Can he wear more than one jersey at a time? etc.)

Problem 17. Fred (shown in blue in Figure 2) lives at the intersection of 1st and A streets. Every day he walks to school (shown in red), which is at the intersection of 5th and E streets. How many different ways can Fred walk to school (assuming he only walks east and north)?

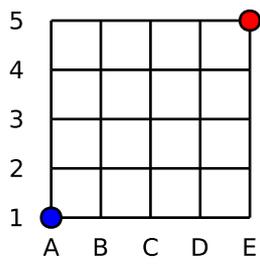


Figure 2: The streets of Manhattan