#21: The Art of Counting, Part II

March 14, 2009

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1 Permutations, revisited

permutations

In last week's assignment, we studied *permutations*, noting that there are $n \cdot (n-1) \cdot (n-2) \ldots 2 \cdot 1 = n!$ (n factorial) different ways to put n objects in order.

Problem 1. How many ways are there to put zero objects in order? What does this suggest the definition of 0! should be?

In general, there are

$$P(n,k) = n \cdot (n-1) \ldots (n-k+1) \quad (1)$$

permutations of k out of n things (that is, the number of ways to put k out of n things in some order). That $n-k+1$ may seem a little mysterious, but if you think about it for a bit, you'll see that all we're doing is taking k numbers, starting from n and counting down by 1 each time. So, the first number starting from n is $n-1+1$ (that is, n); the second number is $n-2+1$; the third number is $n-3+1$, ... and the kth number is $n-k+1$. So, for example, there are twelve ways to put two out of four things in some order: there are 4 possibilities for the first thing, and 3 for the second thing, for a total of $4 \cdot 3 = 12$ permutations.

Problem 2. What is $P(21,6)$? That is, how many permutations are there of 6 out of 21 things?

Problem 3. Which problem from last week's assignment had to do with this general permutations formula?

Problem 4. Calculate 1!, 2!, 3!, ..., 7!. Since factorials come up so often in combinatorics, it's useful to be familiar with the factorials of small numbers.

Problem 5. The number of permutations of k out of n things can also be written as

$$P(n,k) = \frac{n!}{(n-k)!}. \quad (2)$$

Explain why equation (2) is equal to equation (1). (*Hint:* write out the factorials and see what cancels...)

nPr

The number of permutations of k out of n things is implemented on many calculators (probably including yours) as a function called `nPr`. For example, `6 nPr 3` tells you how many ways there are to put three out of six things in some order, that is, the number of permutations of three out of six objects.

2 Combinations

combinations

What if we just want to know the number of ways of choosing a certain number of things from a bigger set of possibilities, but we *don't care* about the order? For example, suppose you are at the store and there are six kinds of fruit to choose from (apples, bananas, cherries, dates, eggplants[1], and figs), but you only have enough money to buy exactly three, and you want to know how many different choices you can make. In this case, the *order* in which you buy your three fruits doesn't make any difference; the only thing that matters is which three fruits you get. A set of objects chosen from some larger set, where we *don't* care about the order of the objects, is called a *combination*.

fruity goodness

Problem 6. So, how many ways *are* there to buy three out of the six fruits? For now, just list the different possibilities (you can abbreviate the fruits using just their first letter). Which three would you buy?

Problem 7. How many *permutations* of three out of six fruits are there? (For example, maybe you plan to buy the fruits one at a time instead of all at once, on Monday, Wednesday, and Friday, so now you care about which order you buy them in.)

Problem 8. Is your answer to Problem 7 bigger or smaller than your answer to Problem 6? Why? How much bigger or smaller is it?

counting combinations

Let's think about how to count the number of ways to choose k out of n things, when we don't care about the order. We already know how to count them when we *do* care about the order, and want to count each ordering separately, but this is *too many*. For example, if we're trying to count the number of ways to choose two out of three things, we could list all the *permutations* of two out of three things—AB, BA, AC, CA, BC, CB—but

[1] Eggplant *is* a fruit. Go look up the definition of fruit if you don't believe me. Botanically speaking, a fruit is anything which is the seed-bearing portion of a plant, regardless of whether it tastes sweet or you could use it to make ice cream. Other "vegetables" which are actually fruits include tomato, squash, pumpkin, corn, cucumber, and zucchini.

this is too many, we've listed each combination of things twice! Since we don't care about the order, AB is really the same as BA, AC is the same as CA, and BC is the same as CB, so there are three ways to choose two out of three things, not six.

combinations from permutations

But herein lies the key. Suppose we are choosing k out of n things. We already know how to count the permutations of k out of n things; as we have noted, this overcounts the combinations, but we can figure out exactly by how much it overcounts. If we count permutations, we count every possible order of each group of k things, but we instead want to just count this group once. Well, how many possible orders are there of k things? Easy—there are $k!$ (that's k factorial, not me being excited about how easy it is). So if we just take the number of permutations and divide by $k!$, we get the number of combinations.

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}. \tag{3}$$

binomial coefficients

Combinations come up so often that they have a special notation: $\binom{n}{k}$, pronounced "n choose k", denotes the number of combinations of k out of n things. $\binom{n}{k}$ is also often called a "binomial coefficient" (for reasons we will see later). You can write it in LaTeX as `\binom{n}{k}`. Your calculator probably has a function to compute combinations called `nCr`.

Problem 9. Compute $\binom{n}{k}$ for every value of n from 0 to 7 and every value of k from 0 to n (that is, compute $\binom{0}{0}$; $\binom{1}{0}$ and $\binom{1}{1}$; $\binom{2}{0}$, $\binom{2}{1}$, and $\binom{2}{2}$; and so on). Make a table with n down the side and k along the top. What patterns do you notice? Can you explain any of the patterns?

Problem 10. Compute $\binom{10}{5}$.

Problem 11. Go back and look at Problem 17 from last week's assignment, and compare it with your table from Problem 9, and your answer to Problem 10. What do you notice? Can you explain the relationship?

3 More problems

Problem 12. How many different poker hands are there? (A poker hand is five cards.)

Problem 13. Remember Fred from Problem 17 in last week's assignment? What if his school was at 7th and K streets? How many ways could he walk to school then?

Problem 14. Fred likes going to Joe's Pizza Parlor, where you can get a Small, Medium, Large, or Ridiculous pizza with your choice of any three toppings. In fact, he likes it so much that he goes once every day and eats an entire pizza. Out of principle, however, Fred never orders a pizza that he has ordered before. (Fred likes Trying New Things.) If Joe's has seventeen toppings to choose from, how long will Fred be able to go to Joe's before he is forced to order a pizza that he has ordered before?

Problem 15. Each day, after his daily pizza, Fred also likes going to the Tastie-Freeze Ice Cream Store and getting a banana split. A banana split consists of three scoops of ice cream (each of the three scoops must be a *different* kind of ice cream) and any two different toppings. Recall that Tastie-Freeze has thirty kinds of ice cream and four toppings. How many different banana splits could Fred order?

Problem 16. What if the three scoops in a banana split do not necessarily have to be different (Fred could get three scoops of the same kind of ice cream, or two of one kind and one of another)? How many different banana splits are there now? (*Hint:* break the types of banana splits down into splits with three different kinds of ice cream, with two kinds, and with one kind, and count each sort of split separately. Keep in mind that the order of the scoops doesn't matter—but getting two scoops of chocolate and one of vanilla IS different than getting two vanilla and one chocolate!)

