#22: Probability
March 22, 2009

Probability, to put it informally, is the mathematical study of *predicting the future*. Given some future event, we often want to know *how likely* it is to happen; probability gives us a way to answer this sort of question. Put another way, probability gives us a way to *quantify uncertainty*. It is a fundamental mathematical tool that has many applications in a wide range of fields.[1] Interestingly, it is also fundamentally connected to combinatorics, so you'll get another chance to exercise your new counting skills!

1 Probability basics

definitions

A *probability* is a real number between 0 and 1. We assign probabilities to *events*. If an event has a probability of 0, that means that the event will certainly *not* happen. Events with a probability of 1 are those which certainly *will* happen. All other events are somewhere in between. An event with probability $1/2$ has a 50% chance of happening, and so on.[2]

We often use the letters p and q to stand for probabilities, and capital letters like A, B, C, X, Y, Z to stand for events. We write $P(A)$ to denote the probability of event A.

Problem 1. Event A has probability p. What is the probability that event A does *not* happen? (For example, if there is an 80% chance of rain tomorrow, what is the chance that it will not rain?)

Fundamentally, we can compute the probability of some event A if we know how to count the number of ways it could happen, and the total number of things that could possibly happen:

$$P(A) = \frac{\text{number of ways } A \text{ could happen}}{\text{total number of things that could possibly happen}}. \qquad (1)$$

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[1]So even if you haven't been paying attention all year, you should definitely pay attention this week...

[2]Actually, I'm sweeping a lot under the rug here, but hopefully you get the intuitive idea.

Another way to say the same thing is

$$P(A) = \frac{\# \text{ ways } A \text{ could happen}}{\# \text{ ways } A \text{ could happen} + \# \text{ ways } A \text{ could } not \text{ happen}}.$$

Since A can either happen or not, the total number of things that can happen must be divided into those scenarios in which A does happen, and those in which it doesn't.

Problem 2. Suppose you have a fair six-sided die[3] (that is, when you roll the die, it is equally likely that any one of the six sides will come up).

    (a) What is the probability of rolling a 1?

    (b) What is the probability of rolling something greater than 2?

a caveat You should keep in mind, however, that this only works if all of the things you are counting are equally likely! For example, it is *not* valid to say, "What's the probability that I will go to the moon tomorrow? Well, there are two things that could happen: either I will go to the moon, or I won't. I end up on the moon in one of those two cases. So the probability that I will go to the moon is 1/2." This isn't valid because there's no reason to believe that the two events you counted (going to the moon and not going to the moon) are equally likely.

Problem 3. Remember Fred? If he randomly picks a way to walk to school each morning (that is, every possible way he could walk to school is equally likely), what is the probability that on a given day he will go through the intersection of first and E streets on his way to school?

Problem 4. What is the probability that Fred will go through the intersection of first and D streets? (*Hint*: think about how many ways there are for Fred to walk from the intersection of first and D to school.)

Problem 5. *Optional extra credit problem:* Forget about street names, and suppose Fred lives on a Cartesian grid, with his house at coordinates $(0,0)$ (hence his school is at $(4,4)$). If Fred randomly picks a way to walk to (x,y) (he is equally likely to pick every possible way), what is the probability that he will pass through (w,z)? You may assume that $w \leq x$ and $z \leq y$. Your answer should be a probability expressed in terms of x, y, w, and z. (Note: just because this is extra credit doesn't mean I won't give you any hints if you ask for them. ⌣)

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[3]The reason there are so many probability problems involving dice and cards is that probability was invented by mathematicians who were also gamblers. In fact, that's why they invented probability. (Really!) Don't try this at home.

2   Combining probabilities

We can often compute the probability of some event as a combination of simpler probabilities.

2.1   Mutually exclusive events

mutual exclusivity   Two events A and B are *mutually exclusive* if they cannot happen at the same time. For example, the two events "the high temperature tomorrow will be 60" and "the high temperature tomorrow will be 55" are mutually exclusive; the high temperature can't be both 60 and 55. But the events "you draw a red card from the deck" and "you draw a nine from the deck" are *not* mutually exclusive, because both could happen: you could draw a red nine.

Problem 6. Which of the following pairs of events are mutually exclusive? Explain your reasoning; don't just say "yes" or "no".

    (a) Randomly running into an old teacher at the mall tomorrow, and randomly running into your cousin at the bowling alley tomorrow

    (b) Rolling a two with a fair six-sided die, and rolling a three

    (c) Dying by getting struck by lightning, and dying by getting eaten by a shark

    (d) Putting peanut butter on your sandwich, and putting pepperoni on your sandwich

    (e) Flipping a coin and having it come up tails, and having it come up heads

$\boxed{P(A \cup B)}$

> If two events A and B are mutually exclusive, then the probability that *either one* will happen, written $P(A \cup B)$, is $P(A) + P(B)$.

Problem 7. Compute the probability of each of the following.

(a) The probability of rolling either a three, a seven, or a fourteen with a fair twenty-sided die.

(b) The probability of flipping a coin and having it come up either heads or tails.

(c) Remember the petting zoo? The one with sixteen rabbits, fourteen snails, and thirty-nine lemurs? Suppose you randomly pick one of the animals to adopt (you are equally likely to pick any one of the animals). What is the probability that you will not pick a snail?

2.2 Independent events

independence

Two events A and B are *independent* if neither one has any effect on the probability of the other. For example, the two events "your pet snail eats some lettuce" and "there will be a power outage" are independent,[4] since whether or not your snail eats lettuce has no bearing on the probability of a power outage, and vice versa. However, the events "there is a thunderstorm" and "there will be a power outage" are not independent: if there *is* a thunderstorm, a power outage is much more likely. Here's another way of thinking about it: if someone asked you, "What's the probability there will be a power outage today?" you might say, "I don't know, maybe 1/100." If they then said, "By the way, there's going to be a thunderstorm this afternoon," you would say, "oh, then the probability of a power outage is higher, maybe 1/20." The fact that the probability of a power outage changed after you knew there was going to be a thunderstorm means that these events are not independent.

Problem 8. Which of the following pairs of events are independent? Explain your reasoning; don't just say "yes" or "no".

(a) Rolling a two with a fair six-sided die, and then rolling a three

(b) Putting peanut butter on your sandwich, and putting pepperoni on your sandwich

(c) Flipping a coin and having it come up tails, and then flipping it and having it come up heads

(d) Seeing an advertisement for Product X, and buying Product X

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[4]Unless your snail has superpowers.

$\boxed{P(A \cap B)}$

> If two events A and B are independent, then the probability that *both* will happen, written $P(A \cap B)$, is the product $P(A)P(B)$.

If two events are *not* independent, then you must not multiply their probabilities! It is possible to analyze the probability of dependent events using something called conditional probability, but we won't get into that today.

Problem 9. Compute the probability of each of the following.

(a) The probability of rolling four 4's in a row with a fair six-sided die.

(b) The probability of flipping ten heads in a row with a fair coin.

(c) You have a drawer with seven pairs of socks; two pairs are red and the rest are blue. In another drawer are some shirts: two white, one yellow, and six green with orange stripes. If you get dressed in the dark, what is the probability that you will end up wearing red socks and a stripey shirt? (Assume that your clothes are all jumbled up in the drawers so you are equally likely to pick any clothing item.)

3   More problems

ask for hints!

Feel free to ask for hints if you get stuck! Some of these are tricky.

Problem 10. Imagine that you flip a fair coin ten times. What is the probability of getting exactly three heads, and all the rest tails?

Problem 11. You are planning to go to the petting zoo again. However, there is a probability of 1/10 that it will rain (and no one likes going to the petting zoo in the rain). Independently, there is also a probability of 2/9 that you will forget that you were planning to go. What is the probability that you will actually go to the petting zoo?

Problem 12. If you draw two cards from the top of a randomly shuffled deck, what is the probability that they will both be aces?

Problem 13. If you draw two cards from the top of a randomly shuffled deck, what is the probability that they will be the same suit?

Problem 14. What is the probability that in a room of fifty people, no two people will share the same birthday? (*Hint:* how many ways are there to pick everyone's birthday? How many ways are there to pick the birthdays so they don't overlap?)