#23: Sequences

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a mysterious rule

Suppose n is an integer, and consider this simple rule: if n is even, divide it by two; otherwise, multiply n by 3, and add one. Pretty simple, right? Let's call it Rule H. Written formally,

$$H(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{otherwise.} \end{cases}$$

easy peasy

Problem 1. What do you get when you apply Rule H to each of the following integers?

(a) 10

(b) 29

(c) 5

(d) 1

(e) 0

(f) -5

iteration

Not all that exciting, huh? Well, consider applying Rule H repeatedly, instead of just once (this is called *iterating* the rule). For example, if we start with 10, applying the rule once gives 5; applying it again to 5 gives 16, applying it to 16 gives... and so on.

Problem 2. Continue iterating Rule H starting with 10. What happens?

Problem 3. What happens when you iterate Rule H starting from each of the following integers?

(a) 3

(b) 6

(c) 7

(d) 9

(e) 15

(f) 27 (*Hint:* this one is mean of me, you don't have to do the whole thing. What do you *think* will happen?)

the Collatz conjecture

As you can see, iterating H leads to some rather unpredictable, crazy behavior! It seems like (most? all?) numbers eventually reach 1 if you iterate H, but some might take a long time to get there. Will *every* starting number eventually reach 1 if you iterate H? Well... *no one knows*!! Most mathematicians think they will—this is called the *Collatz conjecture*, named after the German mathematician Lothar Collatz—but no one has been able to prove it. (But no one has been able to find a counterexample, either.)

1 Sequences

A list of numbers in a particular order is called a *sequence*. For example, iterating Rule H starting from 11 yields the sequence

$$11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, \ldots$$

When we talk about sequences, we usually mean ones that follow particular patterns, although a list of random numbers still technically counts as a sequence.

The numbers in a sequence are called *terms*, and we often use subscript notation to refer to the terms in a sequence: for example, if t is a sequence, then t_1 is the first term in the sequence, t_2 is the second term, and so on.

It can be fun to try to guess the pattern behind a sequence. Did you think that computers are far better than humans at anything having to do with math? Think again! Human brains are incredible pattern-recognizing machines.[1] Computers are great at doing tedious, repetitive calculations without getting tired or making mistakes, but they're only good at following instructions; it's very difficult to reduce pattern recognition to a set of instructions.[2]

[1] In fact, they're so amazing that they tend to notice patterns in places where there actually aren't any. This is why people say that clouds look like llamas (or whatever), and why, if you stare at static on a TV screen long enough, you will start to see lines and shapes emerging from the chaos, and why people claim to see pictures of the Virgin Mary in half-eaten grilled cheese sandwiches.

[2] This is actually an active area of ongoing research in artificial intelligence.

Problem 4. Put your pattern-recognizing machinery to work! Find the pattern or rule behind each sequence and write down the next term of each.

(a) 1, 3, 5, 7, 9, 11, ...

(b) 3, 1, 4, 1, 5, 9, 2, ...

(c) 1, 3, 7, 15, 31, 63, ...

(d) 1, 2, 4, 8, 16, 14, 10, ...

(e) 1, 4, 3, 7, 5, 10, 7, ...

(f) 2, 6, 30, 210, ...

(g) 1, 3, 4, 7, 11, 18, 29, ...

(h) 10, 4, 6, −2, 8, ...

(i) 3, 4, 7, 12, 19, 28, ...

(j) 2, 9, 64, 625, ...

2 Recursive and explicit definitions

recursive definitions

The most natural and fundamental way to define a sequence is with a *recursive* definition (also known as a *recurrence relation*), which uses previous terms to calculate new ones. For example,

$$t_n = 2 \cdot t_{n-1} + 1$$

is a recursive definition which says that the nth term is one more than twice the $(n-1)$st term. We also need some sort of *base case* that tells us where to start—one or more terms must be defined without reference to previous terms. In this case, we might say $t_1 = 1$. So we know to start with 1, which means the next term is $2 \cdot 1 + 1 = 3$, and the next after that is $2 \cdot 3 + 1 = 7$, then 15, and so on. You may notice that the all of the "hailstone sequences" from Problems 2–3 are defined recursively: for example, $t_1 = 10$ and $t_n = H(t_{n-1})$.

Problem 5. Consider the sequence recursively defined by

$$\begin{aligned} t_1 &= 1 \\ t_n &= (t_{n-1})^2 + 1. \end{aligned}$$

What is t_5 (the fifth term in the sequence)?

Problem 6. Consider the following recursive definition:

$$\begin{aligned} t_1 &= 4 \\ t_n &= 5t_{n-2} + 3 \end{aligned}$$

What is t_5? How about t_4? What's the problem? How would you fix it?

explicit definitions

Another way to define a sequence is with an *explicit* definition, which describes how to calculate any term using only its position in the sequence (often denoted n). For example, we might have $s_n = 2^n - 1$. This explicit formula says that to get the nth term in the sequence s, raise 2 to the power of n and then subtract one. So, if n is 1, 2, 3, 4, ... we get 1, 3, 7, 15, ... and so on.

Problem 7. As you may have already guessed, the two definitions used as examples above actually define the same sequence:

$$s_n = 2^n - 1$$

and

$$\begin{aligned} t_1 &= 1 \\ t_n &= 2 \cdot t_{n-1} + 1 \end{aligned}$$

both define the sequence 1, 3, 7, 15, 31, ..., which you met in Problem 4.

(a) Calculate the twentieth term in the sequence. Which definition did you use?

(b) 72057594037927935 is a term in the sequence. Calculate the terms immediately following and preceding it. Which definition did you use?

Problem 8. Let's prove that the sequences s and t in Problem 7 are actually the same (I *said* they were the same, but you shouldn't just take my word for it!).

We can prove that they are the same by *induction*: if we can prove that $t_1 = s_1$, and that whenever $t_{k-1} = s_{k-1}$ then $t_k = s_k$ as well, then we have proved that t_n must be equal to s_n for all values of n (Since $t_1 = s_1$, we know that $t_2 = s_2$; which means that $t_3 = s_3$; which means that $t_4 = s_4$; which means. . .). It's like knocking over dominoes, math-style.

(a) Show that $t_1 = s_1$.

(b) *Assume* that $t_{k-1} = s_{k-1}$, and show that $t_k = s_k$. (*Hint:* start with the definition of t_k, then substitute using the assumption, and then simplify. . .)

3 Some special sequences

3.1 Arithmetic sequences

Any sequence with the following recursive definition is called an *arithmetic*[3] sequence:

$$\begin{aligned} t_1 &= a \\ t_n &= t_{n-1} + d \end{aligned} \tag{1}$$

a and d can be any real numbers; a is the initial term and d is the *common difference*.

Problem 9. Explain, in your own words, what an arithmetic sequence is, and give an example. Why is d called the common difference?

Problem 10. What sequence is obtained when $a = 7$ and $d = -2$? Write the first seven terms.

Problem 11. Is 5, 5, 5, 5, . . . an arithmetic sequence? Give values for a and d, or explain why it is not arithmetic.

Problem 12. Is 1, 4, 9, 16, . . . an arithmetic sequence? Give values for a and d, or explain why it is not arithmetic.

Problem 13. Consider the arithmetic sequence with first term $a = 4$ and common difference $d = 9$. Find t_2, t_5, and t_{193}.

[3] For some odd reason, the noun *arithmetic* (the subject you studied in elementary school) is pronounced uh-RITH-meh-tic, but the adjective *arithmetic* (the type of sequence) is pronounced AIR-ith-MEH-tic. Go figure.

Problem 14. Come up with an explicit definition for the nth term of an arithmetic sequence with initial term a and common difference d.

3.2 Geometric sequences

Any sequence with the following recursive definition is called a *geometric* sequence:

$$\begin{aligned} t_1 &= a \\ t_n &= r \cdot t_{n-1} \end{aligned} \tag{2}$$

a is the initial term and r is the *common ratio*.

Problem 15. Explain in your own words what a geometric sequence is, and give an example. Why is r called the *common ratio*?

Problem 16. What sequence is obtained when $a = 16$ and $r = 3/2$? Write the first seven terms.

Problem 17. Consider the geometric sequence with first term $a = 5$ and common ratio $r = 2$. Find t_2 and t_{15}.

3.3 The Fibonacci sequence

Consider the sequence defined recursively as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{when } n \geq 2. \end{aligned}$$

This is known as the *Fibonacci sequence*; it is one of the most famous sequences in all of mathematics. (Note that the recursive definition for the Fibonacci numbers specifies *two* base cases, since in the general case each term is defined as the sum of the *two previous* terms; you learned why this is necessary in Problem 6.)

Problem 18. Compute the first fifteen terms of the Fibonacci sequence, beginning with F_0.

Problem 19. Add up the first two terms of the Fibonacci sequence, then add up the first three terms, then the first four, then five, and so on. What do you notice? (*Hint:* try adding one to each of the sums...)

Problem 20. Visit the On-Line Encyclopedia of Integer Sequences (just Google the name, you'll find it). This is one of my favorite websites; it has more information than you could ever want to know on more integer sequences than you could ever imagine.

(a) Search for the Fibonacci sequence by typing the first seven or so terms into the search box. A page should come up with a ton of information about the Fibonacci sequence. You probably won't understand all of it, but that's OK; I don't either. Find something interesting that you *do* understand and write about it.

(b) Now go to the very bottom of the page and click on "WebCam". Change "reload every 20 seconds" to "when I say so". Now click on "Next sequence" until you find an interesting sequence that you understand (note: even if you don't understand all the words in the main definition of the sequence, there are often further explanation and examples further down on the page). There may be quite a few that are defined in terms of things you don't know; that's OK, just keep skipping until you find one that makes sense. Write about it.

