#30: Graph theory

May 25, 2009

Graph theory is the study of *graphs*. But *not* the kind of "graphs" you are used to, like a graph of $y = x^2$—graph theory graphs are completely different from graphs of functions.[1]

Informally, graphs describe *connections* or *networks*. But before we say formally what a graph is, let's look at an example—in fact, *the* example which got graph theory started!

1 The seven bridges of Königsberg

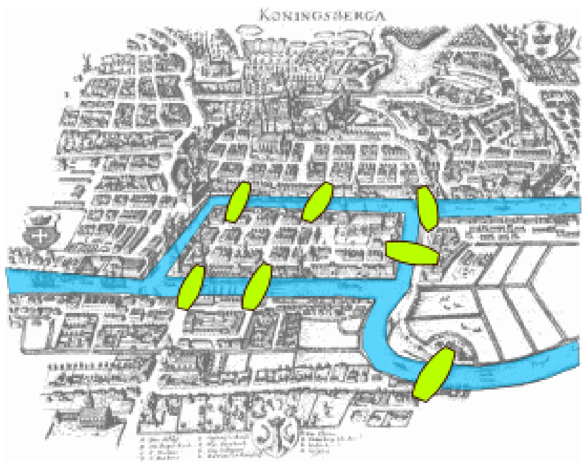


Figure 1: The bridges of Königsberg

The city of Kaliningrad, Russia, is built on both sides of the Pregel river, and there are two large islands in the middle of the river. Prior to World War II, the city was called Königsberg, and there were seven bridges connecting the islands to the rest of the city, as shown in Figure 1. (The bridges are much different now; some were destroyed in World War II and some were knocked down to make room for a new highway.) The story goes that it

[1] Mathematicians seem to have a bad habit of using the same word for several different things. Confusing, isn't it?

was a popular riddle to ask if there was any way to go on a walking tour of the city in which each bridge was crossed *exactly once*. Tricky things like crossing a bridge halfway are not allowed.

Problem 1. It turns out that it isn't possible, so I won't make you try to solve it. But can you make it possible by adding a single bridge? Describe where the new bridge could be, and then describe a walking tour that crosses each bridge (including the new one) exactly once.

Euler and graph theory

Euler (remember him?) solved the problem of the bridges in 1735, and in so doing invented the subject of graph theory.

Euler's stroke of genius was to clear away all the detail that didn't matter and consider the problem in an abstracted setting. The fact that the walking tour takes place in a physical city doesn't matter; for the purposes of the problem, the only things that matter are that there are four locations (the north bank, the south bank, the west island, and the east island) and certain connections between them (the bridges). Schematically, we can draw the situation as shown in Figure 2.

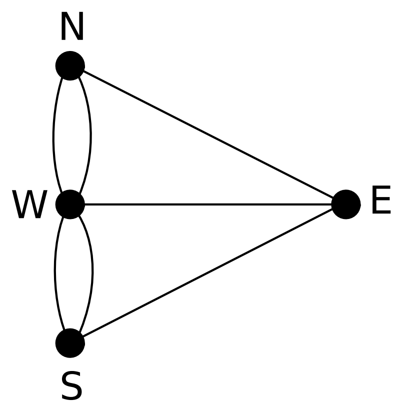


Figure 2: A graph representing the bridges of Königsberg

N and S represent the north and south banks of the river; W represents the west island and E represents the east island. We represent each separate land area by a dot, and for each bridge we draw a line between the dots representing the land areas that it connects.

Of course, we haven't really changed anything; but you'd be surprised how much it helps the human brain to clear away irrelevant detail! By making a diagram representing only the relevant information (how the four land areas are connected by bridges) it becomes a lot easier to think about the problem.

If you try solving the Königsberg bridge problem, you'll quickly find that you're getting nowhere—you always seem to end up with one bridge left to cross but in the totally wrong place. But this isn't proof that there's no solution—maybe you just haven't been clever enough to come up with the solution yet! How was Euler able to be so sure that there was no solution?

We'll get to that, but first let's be a bit more formal about what a graph is.

2 Graphs

graph = vertices + edges

Formally, a graph consists of a collection of *vertices* (also called *nodes*), together with a collection of *edges*. Each edge connects two vertices. The vertices usually are given names so that you can tell them apart.

geometry? forget it

The particular way in which a graph is drawn does not matter; the only thing that matters is which vertices are connected by edges. For example, these two graphs are really the same:

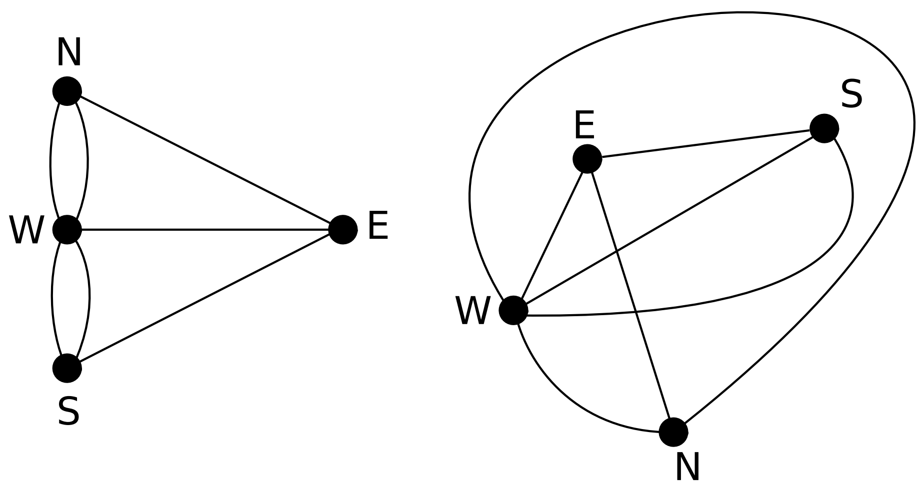
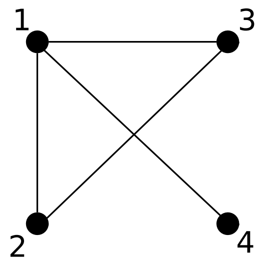


Figure 3: Two different drawings of the same graph

In other words, these are just two different ways to draw the *same graph*, since they have the same vertices and the same connections between vertices. Notice that some of the edges cross each other in the drawing on the right, but that's perfectly OK.

Problem 2. Which of the following drawings represent the same graph?



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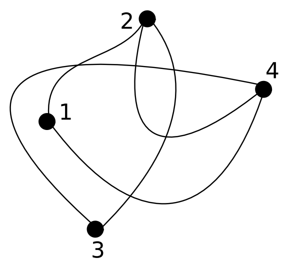

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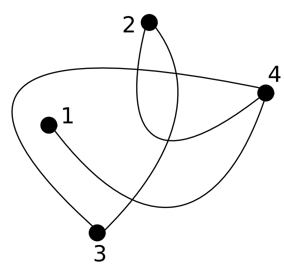

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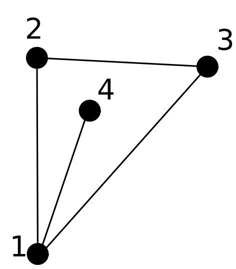

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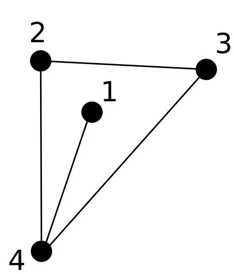


(e)

2.1 Vertex degree

degree

The *degree* of a vertex is the number of edges that are connected to it.

Problem 3. List the degree of each of the vertices in the graph from part (a) of Problem 2.

Problem 4. List the degree of each of the four land areas in Königsberg.

Problem 5. Graph G has five vertices, and all of the vertices have degree 2. What does G look like?

3 Paths

path

A *path* in a graph is a sequence of edges whose vertices match up; that is, a path is any journey that an ant could take as it walks along edges in the graph. The *length* of a path is the number of edges in it.

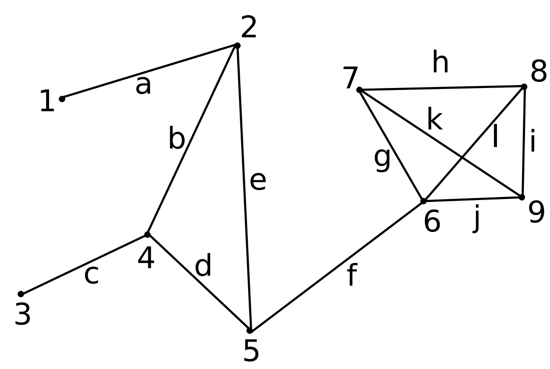


Figure 4: An example graph with labeled vertices and edges

For example, consider the graph in Figure 4. The edges are labeled so we can easily talk about them. One example path in this graph from vertex 1 to vertex 8 is the sequence of edges a-e-f-g-h. Note that paths can do weird things like reverse direction, loop back on themselves, and so on—for example, a-e-d-b-e-f-l is another path from 1 to 8; so is a-e-e-e-f-l (the ant walked back and forth on edge e three times, from 2 to 5, then back to 2, and then back to 5).

Note that the graph in Figure 4 doesn't have multiple edges between any vertices, so we can also describe paths by listing the vertices that the path visits in order: for example, the path a-e-e-e-f-l can also be described as 1-2-5-2-5-6-8. Of course, if a graph has multiple edges between vertices (as, for example, the Königsberg bridge graph) then you have to actually specify the edges themselves, because otherwise you wouldn't know which edges to use.

Problem 6. Write down a path from vertex 9 to vertex 3.

Problem 7. Write down a path from vertex 8 to vertex 4 which has length 12.

cycle

A *cycle* is a path which begins and ends at the same vertex.

Problem 8. Write down a cycle in the above graph that starts and ends at vertex 3.

3.1 Eulerian paths

An *Eulerian* path, named in honor of Euler, is a path which includes each edge in a graph exactly once. Of course, this is exactly what the Königsberg bridge problem is about: the problem is to find a walking tour (path) that crosses every bridge (edge) exactly once. That's why this kind of path is named after Euler. So, the Königsberg bridge problem can be rephrased as, "Is there an Eulerian path in the Königsberg bridge graph?"

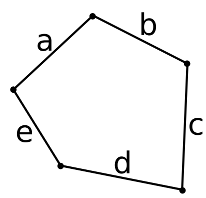
Interestingly, deciding whether a graph has an Eulerian path is very easy, as shown by Euler: a graph G has an Eulerian path if and only if

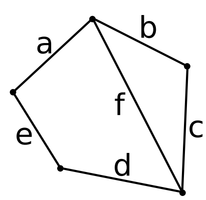
- every vertex in the graph has an even degree, OR
- exactly two vertices have an odd degree, and all other vertices have an even degree.

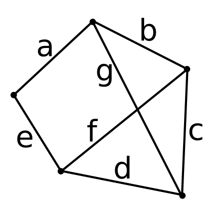
Why is this? Well, think about visiting some vertex in a graph in the middle of an Eulerian path. You come into the vertex along some edge, and leave along another edge (since you can't use the same edge twice). So as long as the vertex has even degree you are OK—every time you enter the vertex along an unused edge there will be another unused edge along which you can leave. If some vertex has an odd degree then you will get stuck, since at some point there will be exactly one edge left, and you will be able to enter the vertex but not leave. The only exception is if that vertex is at the beginning or the end of the path, which is why it's OK to have exactly two vertices with odd degree—those will be the starting and ending points of the path.

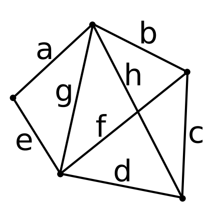
Now that we know this, it's easy to see that the Königsberg bridge graph is not Eulerian: all four of its vertices have odd degree, which is too many!

Problem 9. For each graph, write down an Eulerian path, or state that no such path exists.

(a) 

(b) 

(c) 

(d) 

3.2 Hamiltonian paths

A *Hamiltonian* path, named in honor of the mathematician William Rowan Hamilton, is a path which visits each *vertex* exactly once. Note that it does not matter whether a Hamiltonian path visits all the edges in a graph.

Interestingly, Hamiltonian paths are *much* harder to find than Eulerian paths! There is no efficient way known to tell whether any given graph has a Hamiltonian path or not, although it is easier for certain graphs.

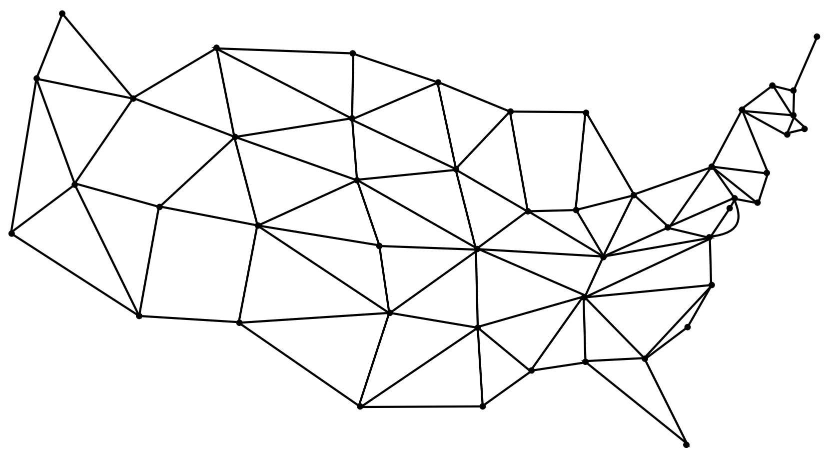


Figure 5: Mystery graph

Problem 10. What does the graph in Figure 5 represent? What are the nodes? What are the edges?

Problem 11. Does the graph in Figure 5 have a Hamiltonian path? If so, write it down; if not, explain how you know there isn't one. Explain the significance of your answer as it relates to epic roadtrips.

