Lecture 4: Transfinite recursion, cardinals

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Definition 4.16. The class of *sequences* Seq is defined by

$$Seq = \{\, f \mid \mathrm{ord}(\mathrm{dom}(f)) \wedge f \text{ is a function} \,\}.$$

Theorem 4.17 (Transfinite Recursion)**.** *For any functional relation $G : Seq \to V$, there exists a unique functional relation F satisfying*

$$F(\alpha) = G(F \upharpoonright \alpha)$$

for all α.

Proof. We will show that for every α there is a unique function f_α such that $\mathrm{dom}(f_\alpha) = \alpha$, and $\forall \beta < \alpha$,

$$f_\alpha(\beta) = G(f_\alpha \upharpoonright \beta),$$

and $\forall \gamma < \beta$, $f_\beta \upharpoonright \gamma = f_\gamma$.

The proof is by transfinite induction.

- $\alpha = 0$. $f_\alpha = \{\}$ trivially satisfies the conditions.
- $\alpha = \beta + 1$. By the IH, assume there exists a unique f_β that satisfies the conditions. Now let

  $$f_{\beta+1}(\gamma) = \begin{cases} f_\beta(\gamma) & \gamma < \beta \\ G(f_\beta) & \gamma = \beta. \end{cases}$$

  We must show that for every $\delta < \beta + 1$, $f_{\beta+1}(\delta) = G(f_{\beta+1} \upharpoonright \delta)$. There are two cases.

  - If $\delta = \beta$, then $f_{\beta+1}(\delta) = G(f_\beta) = G(f_{\beta+1} \upharpoonright \beta)$, since it is clear from the definition of $f_{\beta+1}$ that $f_{\beta+1} \upharpoonright \beta = f_\beta$.
  - If $\delta < \beta$, then $f_{\beta+1}(\delta) = f_\beta(\delta)$, which is equal to $G(f_\beta \upharpoonright \delta)$ by the IH. But this is equal to $G(f_{\beta+1} \upharpoonright \delta)$ by definition of $f_{\beta+1}$.

  By the IH, we already know that $f_\beta \upharpoonright \zeta = f_\zeta$ for all $\zeta < \beta$; we must show that forall $\zeta < \beta + 1$, $f_{\beta+1} \upharpoonright \zeta = f_\zeta$. First, if $\zeta < \beta$, this follows from the IH and the definition of $f_{\beta+1}$. If $\zeta = \beta$, we must show $f_{\beta+1} \upharpoonright \beta = f_\beta$; this follows immediately from the definition of $f_{\beta+1}$.

  The last thing we must show is that $f_{\beta+1}$ is unique. Suppose there is some h which also satisfies the conditions, that is, $\mathrm{dom}(h) = \beta + 1$ and $\forall \delta < \beta + 1, h(\delta) = G(h \upharpoonright \delta)$. Then pick $\delta < \beta + 1$ to be the smallest ordinal for which $h(\delta) \neq f_{\beta+1}(\delta)$. Then $f_{\beta+1} \upharpoonright \delta = h \upharpoonright \delta$, so $f_{\beta+1}(\delta) = G(f_{\beta+1} \upharpoonright \delta) = G(h \upharpoonright \delta) = h(\delta)$, a contradiction.

- $\lim(\alpha)$. By the IH, assume that for all $\beta < \alpha$, there exists a f_β satisfying the conditions. Then let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

  First, we must show that f_α is a set. This follows from the Axiom of Replacement, since it is the union of the image of α under the map $\beta \mapsto f_\beta$, which is a functional relation by the uniqueness of f_β under the IH.

  The fact that f_α is functional follows from the IH, since we know that $f_\beta \restriction \gamma = f_\gamma$ for all $\gamma < \beta$.

  Let $\beta < \alpha$. Then

$$\begin{aligned} f_\alpha(\beta) &= f_{\beta+1}(\beta) && \beta+1<\alpha, \text{def. of } f_\alpha \\ &= G(f_{\beta+1} \restriction \beta) && \text{IH} \\ &= G(f_\alpha \restriction \beta) && \text{intuitively obvious...?} \end{aligned}$$

  We need do nothing to establish that $\forall \gamma < \beta < \alpha$, $f_\beta \restriction \gamma = f_\gamma$; it already holds by the inductive hypothesis.

  The argument for the uniqueness of f_α is the same as in the previous case.

Now define $F(\alpha) = G(f_\alpha)$. We claim that F satisfies the theorem. Note that F is a functional relation since we have defined it pointwise. Note also that $F \restriction \alpha$ is a set (by Replacement: $F \restriction \alpha = \{(\beta, F(\beta)) \mid \beta \in \alpha\}$). To see that $f_\alpha = F \restriction \alpha$, consider any $\beta \in \text{dom}(f_\alpha) = \text{dom}(F \restriction \alpha) = \alpha$; we have $f_\alpha(\beta) = G(f_\alpha \restriction \beta) = G(f_\beta) = F(\beta)$. ∎

5 Cardinals

Definition 5.1. X is *equivalent* to Y, denoted $X \sim Y$ (or $|X| = |Y|$), if there is a mapping $f : X \xrightarrow[\text{onto}]{\text{1-1}} Y$.

Definition 5.2. $X \leq Y$ if there is a mapping $f : X \xrightarrow{\text{1-1}} Y$.

Theorem 5.3 (Cantor-Schröder-Bernstein)**.** $X \leq Y \wedge Y \leq X \implies X \sim Y$.

Proof. Suppose $f : X \xrightarrow{\text{1-1}} Y$ and $g : Y \xrightarrow{\text{1-1}} X$ are functions implied by the premises. Let

$$\begin{aligned} X_0 &= X - g(Y) \\ X_{n+1} &= (g \circ f)(X_n) \\ X_\omega &= \bigcup_{n\in\omega} X_n. \end{aligned}$$

and define

$$h(a) = \begin{cases} f(a) & a \in X_\omega \\ g^{-1}(a) & a \in X - X_\omega. \end{cases}$$

Note that h is total, since if $a \in X - X_\omega$, then $a \notin X_0$, so $a \in \mathrm{rng}(g)$ and $g^{-1}(a)$ is defined.

We claim that h is a one-to-one, onto function from X to Y.

- To show that h is one-to-one, suppose $a, b \in X$ and $h(a) = h(b)$. If $a, b \in X_\omega$, then $f(a) = f(b)$, so $a = b$ since f is one-to-one. If $a, b \notin X_\omega$, then $g^{-1}(a) = g^{-1}(b)$; applying g to both sides yields $a = b$. So, without loss of generality, suppose $a \in X_\omega$ and $b \notin X_\omega$, $f(a) = g^{-1}(b)$; we claim this case is impossible. Applying g to both sides yields $g(f(a)) = b$; but since $a \in X_\omega$ then b is also, a contradiction.
- Now we show h is onto. Let $b \in Y$, and let $f(X_\omega) = Y_\omega$. If $b \in Y_\omega$, then it is in the image of h, since $h(X_\omega) = f(X_\omega) = Y_\omega$. Otherwise, consider $g(b)$. $g(b) \notin X_\omega$; if it were, $g(b) \in X_n$ for some n, so we would have $g(b) = g(f(q))$ for some $q \in X_{n-1}$. But since g is one-to-one, this implies $b = f(q)$, that is, $b \in Y_\omega$, a contradiction. Therefore, $h(g(b)) = g^{-1}(g(b)) = b$.

∎

Definition 5.4. $X < Y$ if $X \leq Y$ and $Y \not\leq X$.

Theorem 5.5 (Cantor diagonal)**.** *For every X, there exists a Y such that $X < Y$.*

Proof. Claim: $X < \mathcal{P}(X)$. Let $f : X \to \mathcal{P}(X)$, and define

$$a = \{\, b \in X \mid b \notin f(b) \,\}.$$

Note that $a \in \mathcal{P}(X)$. We claim that $a \notin \mathrm{rng}(f)$. If it were, there would be some $c \in X$ with $f(c) = a$; is $c \in f(c)$? If it is, it isn't; if it isn't, it is. So there. f is not onto.

Note that $X \leq \mathcal{P}(X)$, since $f(a) = \{a\}$ is a one-to-one mapping.

If $\mathcal{P}(X) \leq X$, by Cantor-Schröder-Bernstein there would be a one-to-one, onto map between them, but we have shown that any mapping $X \to \mathcal{P}(X)$ is not onto. Therefore, $X < \mathcal{P}(X)$. ∎

Remark. Why is this called a *diagonal* argument? Note that $\mathcal{P}(X) \sim 2^X$ (where X^Y, also sometimes written ${}^{Y}X$, denotes the set of functions from Y to X). In particular, if $Z \subseteq X$, we set $Z \in \mathcal{P}(X)$ to the indicator function

$$g_Z(a) = \begin{cases} 1 & a \in Z \\ 0 & a \notin Z. \end{cases}$$

In the special case that $X \sim \omega$, if we assume there exists a 1-1, onto mapping from X to $\mathcal{P}(X)$, we can make a table of the indicator functions to which each element of X is sent, as follows:

	x_0	x_1	x_2	x_3	$\dots$
x_0	1	0	1	1	
x_1	0	1	0	1	
x_2	0	0	0	1	
x_3	1	0	0	0	
$\vdots$					$\ddots$

The ith row is the indicator function describing the subset to which x_i is sent. Now we simply note that the argument in the above proof corresponds to picking out the diagonal elements (here $1, 1, 0, 0, \dots$), flipping them $(0, 0, 1, 1, \dots)$, and noting that the resulting sequence cannot be a row of the table.

Definition 5.6. κ is a *cardinal* iff κ is an ordinal such that $\alpha \not\sim \kappa$ for all $\alpha \in \kappa$.

Remark. A cardinal κ is an *initial ordinal*—the smallest ordinal having its cardinality.

Exercise: show that every natural number is a cardinal, and that ω is a cardinal (ω is the first infinite cardinal).

Remark. By Theorem 5.5, we know that $\omega < \mathcal{P}(\omega)$. A natural question arises: is there some $X \subseteq \mathcal{P}(\omega)$ for which $\omega < X < \mathcal{P}(\omega)$? This is an interesting question, especially given that it can be shown that $\mathbb{R} \sim \mathcal{P}(\omega)$. Hilbert thought this question so important that he made it the very first problem in his famous 1900 list.

Cantor hypothesized that there does not exist such an X; this hypothesis is known as the *continuum hypothesis* (CH). This is a reasonable hypothesis, especially given the establishment of various special cases, such as the fact that for all $X \subseteq \mathbb{R}$, if X is closed, then it is not the case that $\omega < X < \mathbb{R}$ (Cantor-Bendixson).

It turns out that the continuum hypothesis is independent of ZF: Gödel in 1939 showed that the consistency of ZF implies the consistency of ZF + AC + CH; but Cohen showed in 1963 that the consistency of ZF also implies the consistency of ZF + AC + $\neg$ CH.