Lecture 7: The Real Line
February 9, 2009

6 The Real Line

Definition 6.1. Let $(\mathbb{Q}, <)$ denote the rational numbers with the usual ordering. We define δ to be a formula of first-order logic which expresses the fact that $\mathbb{Q}$ is a dense linear order without endpoints. (Actually translating this into first-order logic is left as an exercise for the reader.)

Definition 6.2. A *partial isomorphism* of orders is a map which is an isomorphism of its domain and range. That is, $f : C \to D$ is a partial isomorphism if for every $e, e' \in C$, $e \leq_C e' \implies f(e) \leq_D f(e')$ whenever $e, e' \in \mathrm{dom}(f)$.

Definition 6.3. A set P of maps from C to D has the *back-and-forth property* iff

- For every $f \in P$ and $c \in C$, there is some $g \in P$ such that $f \subseteq g$ and $c \in \mathrm{dom}(g)$. (This is the "forth" part.)
- For every $f \in P$ and $d \in D$, there is some $g \in P$ such that $f \subseteq g$ and $d \in \mathrm{rng}(g)$. (You guessed it, the "back" part.)

Definition 6.4. C and D are *partially isomorphic*, denoted $C \cong_P D$ iff there is a nonempty set P of partial isomorphisms between C and D which has the back-and-forth property.

Remark. Note that the existence of a partial isomorphism between C and D does *not*, by itself, imply that C and D are partially isomorphic.

Lemma 6.5. *If $C, D \models \delta$, then $C \cong_P D$.*

Proof. Define P to be the set of order-preserving maps f for which $\mathrm{dom}(f)$ is finite, $\mathrm{dom}(f) \subseteq C$, and $\mathrm{rng}(f) \subseteq D$.

P is nonempty, because any singleton map from some element $c \in C$ to any element $d \in D$ is trivially order-preserving.

To see that P has the "forth" property, suppose $f \in P$ and $c \in C - \mathrm{dom}(f)$. Now suppose $c < \min(\mathrm{dom}(f))$, which exists since $\mathrm{dom}(f)$ is finite. Then, since D has no endpoints, there exists some $d \in D$ for which $d < f(\min(\mathrm{dom}(f)))$. Take $g = f \cup (c, d)$; g is order-preserving so $g \in P$. The case when $c > \max(\mathrm{dom}(f))$ is similar. Otherwise, let c_1 be the greatest element of $\mathrm{dom}(f)$ less than c, and c_2 the least element of $\mathrm{dom}(f)$ greater than c; since D is dense, there is some $d \in D$ for which $f(c_1) < d < f(c_2)$. Again, take $g = f \cup (c, d)$; then $g \in P$.

The proof that P has the "back" property is similar, and uses the fact that $C \models \delta$. ∎

Remark. We note that there are partially isomorphic orders which are not isomorphic—in particular, by the previous lemma, $\mathbb{Q} \cong_P \mathbb{R}$, but we know $\mathbb{Q} \ncong \mathbb{R}$, since they have different cardinality.

Theorem 6.6 (Cantor's back-and-forth theorem)**.** *If the orders C and D are partially isomorphic and* $\mathrm{card}(C) = \mathrm{card}(D) = \aleph_0$*, then $C \cong D$.*

Remark. By saying C is an order, we mean it is a pair $\langle X, <_C \rangle$, and define $\mathrm{card}(C) = \mathrm{card}(X)$.

Proof. Let P be the set of partial isomorphisms witnessing the fact that C and D are partially isomorphic. Since C and D are countable, we may enumerate them as

$$C = \{c_0, c_1, c_2, \dots\}$$
$$D = \{d_0, d_1, d_2, \dots\}.$$

Pick any $f_{-1} \in P$, and enlarge it to f_0 such that $c_0 \in \mathrm{dom}(f_0)$; $f_0 \in P$ since P has the forth property.

Now we choose $f_1, f_2, \dots \in P$ as follows. At stage $2n+1$, pick f_{2n+1} to extend f_{2n} with $d_n \in \mathrm{rng}(f_{2n+1})$; at stage $2n+2$, pick f_{2n+2} to extend f_{2n+1} with $c_n \in \mathrm{dom}(f_{2n+2})$.

Finally, let $f = \bigcup_{i \in \omega} f_i$. f is a function, since $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$. Also, $\mathrm{dom}(f) = C$ and $\mathrm{rng}(f) = D$ by construction. Finally, f is order-preserving, since if $c_i <_C c_j$, then $c_i, c_j \in \mathrm{dom}(f_{2\max(i,j)})$, and all the f_k are order-preserving. Therefore, f is an isomorphism. ∎

Corollary 6.7. *For all orders A and B, if* $\mathrm{card}(A) = \mathrm{card}(B) = \aleph_0$ *and $A \models \delta$ and $B \models \delta$, then $A \cong B$.*

Proof. This follows immediately from Lemma 6.5 and Theorem 6.6. ∎

Remark. Note that there are $C, D \models \delta$ where $\mathrm{card}(C) = \mathrm{card}(D) = 2^{\aleph_0}$ but $C \ncong D$. For example, take $C = \mathbb{R}$ and $D = \mathbb{R} - (Irr \cap [0,1])$, where Irr denotes the set of irrational numbers. So δ only categorizes sets of cardinality $\aleph_0$.

Exercise: show that for every $\kappa > \aleph_0$, there are 2^κ pairwise non-isomorphic orders A of cardinality κ ???

Definition 6.8. For a language L, we write $A \equiv_L B$ to mean "A and B can't be distinguished by sentences of L," that is, for all $\varphi \in L$, $A \models \varphi \iff B \models \varphi$.

Remark. $L_{\infty\omega}$ is the maximal language one gets by allowing application of boolean operations ($\bigwedge$, $\bigvee$) to sets of first-order formulas. In other words, $L_{\infty\omega}$ allows infinite conjunction and disjunctions. In general, $L_{\kappa\omega}$ is the language which allows taking the conjunction or disjunction of sets of formulas up to cardinality κ.

Theorem 6.9 (Karp)**.** *If $A \cong_P B$ then $A \equiv_{L_{\infty\omega}}$.*

Remark. The proof is omitted. We note that this immediately implies that $\mathbb{Q} \equiv_{L_{\infty\omega}} \mathbb{R}$! So we need better tools to distinguish $\mathbb{Q}$ from $\mathbb{R}$. What is true about $\mathbb{R}$ that isn't true about $\mathbb{Q}$?

- $\mathbb{R}$ is *order-complete*; that is, every nonempty bounded set of reals has a least upper bound. This is clearly not true about $\mathbb{Q}$, as noted by the ancient Greeks.
- $\mathbb{R}$ is *seperable*, that is, there exists a countable subset which is dense in $\mathbb{R}$ (for example, $\mathbb{Q}$).

So, let γ denote the sentence whose interpretation is "$\mathbb{R}$ is a complete, separable, dense linear order without endpoints."

We can express γ in second-order logic. In particular, to express the fact that a predicate X corresponds to a countable subset of its domain, we can write

$$\exists S.\text{S is 1-1 and almost onto on } X, \text{ and } X, S \text{ satisfies induction.}$$

where "almost onto" means that $|X - \text{rng}(S)| = 1$, and "X, S satisfies induction" means that

$$\forall Y.Y(0) \wedge (Y(n) \wedge S(n,m) \implies Y(m)) \implies (\forall n.X(n) \implies Y(n)).$$

Theorem 6.10. *If $A, B \models \gamma$, then $A \cong B$.*

Proof. Let $\mathbb{Q}^A$ and $\mathbb{Q}^B$ be countable, linearly ordered subsets dense in A and B, respectively. Since $\mathbb{Q}^A$ and $\mathbb{Q}^B$ are dense in A and B, they are dense as well. Also, since A and B have no endpoints, neither do $\mathbb{Q}^A$ and $\mathbb{Q}^B$. Then $\mathbb{Q}^A, \mathbb{Q}^B \models \delta$, and by Corollary 6.7, $\mathbb{Q}^A \cong \mathbb{Q}^B$.

Now, for every $a \in A$, form the set

$$\text{lc}(a) = \{\, b \in \mathbb{Q}^A \mid b <_A a \,\}.$$

(lc(a) corresponds to the lower Dedekind cut for a.) Then define

$$\text{DC}(A) = \{\, \text{lc}(a) \mid a \in A \,\},$$

ordered by $\subseteq$. Then we claim that $\langle A, < \rangle \cong \langle \text{DC}(A), \subseteq \rangle \cong \langle \text{DC}(B), \subseteq \rangle \cong \langle B, < \rangle$.

First, note that lc is an isomorphism from $(A, <)$ to $\langle \text{DC}(A), \subseteq \rangle$.

Now we must exhibit an isomorphism between $\langle \text{DC}(A), \subseteq \rangle$ and $\langle \text{DC}(B), \subseteq \rangle$. Let $f : \mathbb{Q}^A \to \mathbb{Q}^B$ be an isomorphism. Then define a map $F : \text{DC}(A) \to \text{DC}(B)$ which sends X to $f[X]$. We must show that F is well-defined: it is not immediate that $f[\text{lc}(a)] \in \text{DC}(B)$. Note that there must be some $a' \in \mathbb{Q}^A$ greater than a. Moreover, since f is order-preserving, $f(a')$ is an upper bound of $f[\text{lc}(a)]$. Therefore, since B is order-complete, there exists a least upper bound $b \in B$ of $f[\text{lc}(a)]$. We claim that $f[\text{lc}(a)] = \text{lc}(b)$. First, if $x \in \text{lc}(a)$, then $f(x) \in \text{lc}(b)$ since $\text{lc}(b)$ contains all elements of $\mathbb{Q}^B$ less than b. If $y \in \text{lc}(b)$, then there must be some $x \in \mathrm{lc}(a)$ for which $f(x) = y$; otherwise, since f is onto, there would have to be some $x' \geq a$ for which $f(x') = y$, but this would contradict the fact that f is order-preserving.

F is order-preserving since $X \subseteq Y \implies f[X] \subseteq f[Y]$.

We can similarly define $F^{-1} : \mathrm{DC}(B) \to \mathrm{DC}(A)$ which sends X to $f^{-1}[X]$; a parallel argument shows that F^{-1} is well-defined and order-preserving.

Finally, we note that since f is an injection, $f^{-1}[f[X]] = X$, so F and F^{-1} are inverse, and therefore F is an isomorphism. ⌧

Remark. lc in the preceding proof is an injection from $\mathbb{R}$ to $\mathcal{P}(\mathbb{Q})$; therefore, $\mathrm{card}(\mathbb{R}) \leq 2^{\aleph_0}$.

Definition 6.11 (Cantor set)**.** Let $C = \{0, 2\}^\omega$. Then $|C| = 2^{\aleph_0}$.

Now for each $f \in C$, form the sum

$$\sum_{i=1}^{\omega} f(i) \cdot 3^{-i}.$$

This gives the set of real numbers whose "trinary" expansions omit the digit 1.

Remark. We can also construct this set by taking $D_0 = [0, 1]$, D_1 to be D_0 without the middle 1/3, D_2 to be D_1 with the middle 1/3 removed from each of its subintervals, and so on recursively. Then $C = \bigcap_{n \in \omega} D_n$.

Note that C is a closed set with maximal cardinality which is nowhere dense!

If each element of C defines a distinct real number, then we see that $2^{\aleph_0} \leq \mathrm{card}(\mathbb{R})$. Since we showed in the proof of Theorem 6.10 that $\mathrm{card}(\mathbb{R}) \leq 2^{\aleph_0}$, in fact $\mathrm{card}(\mathbb{R}) = 2^{\aleph_0}$.

Definition 6.12. A subset of $\mathbb{R}$ is *open* if it is a union of open intervals. A subset is *closed* if it is the complement of an open set.

Remark. Open sets form a *topology* on $\mathbb{R}$, since they include $\mathbb{R}$ and $\emptyset$ and are closed under arbitrary unions and finite intersections.

Note that $\mathbb{R}$ has a countable basis, namely, the set of open intervals with rational endpoints.

Remark. Consider $|\mathcal{P}(\mathbb{R})| = 2^{2^{\aleph_0}} > 2^{\aleph_0}$. That's a lot of sets! The CH states that every element of $\mathcal{P}(\mathbb{R})$ is either countable or has the same cardinality as $\mathbb{R}$, but it seems difficult to get a handle on something quantifying over such a large set. Perhaps we can make better progress if we look at simpler classes of subsets of $\mathbb{R}$, for example, open sets. There are only $2^{\aleph_0}$ open sets, since each is a countable union of intervals from the countable basis of $\mathbb{R}$.