Lecture 8: The Real Line, Part II

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Definition 6.13. $a \in X$ is *isolated in* X iff there is an open interval I for which $X \cap I = \{a\}$. Otherwise, a is a *limit point.*

Remark. Another way to state this is that a is isolated if it is not a limit point of X.

Definition 6.14. X is a *perfect set* iff X is closed and has no isolated points.

Remark. This definition sounds nice and tidy, but there are some very strange perfect sets. For example, the Cantor set is perfect, despite being nowhere dense!

Our goal will be to prove the Cantor-Bendixson theorem, *i.e.* the perfect set theorem for closed sets, that every closed uncountable set has a perfect subset.

Lemma 6.15. *If P is a perfect set and I is an open interval on $\mathbb{R}$ such that $I \cap P \neq \emptyset$, then there exist disjoint closed intervals $J_0, J_1 \subset I$ such that* $\text{int}[J_0] \cap P \neq \emptyset$ *and* $\text{int}[J_1] \cap P \neq \emptyset$. *Moreover, we can pick J_0 and J_1 such that their lengths are both less than any $\epsilon > 0$.*

Proof. Since P has no isolated points, there must be at least two points $a_0, a_1 \in I \cap P$. Then just pick $J_0 \ni a_0$ and $J_1 \ni a_1$ to be small enough. ∎

Lemma 6.16. *If P is a nonempty perfect set, then $P \sim \mathbb{R}$.*

Proof. We exhibit a one-to-one mapping $G : 2^\omega \to P$.

Note that 2^ω can be viewed as the set of all infinite paths in a full, infinite binary tree with each edge labeled by 0 or 1. We label each node in the tree by the sequence of labels on the path from the root to the node.

Now we associate an interval I_s to each node s, with the properties that

- I_s is closed,
- $I_s \cap P \neq \emptyset$,
- $I_{s,b} \subset I_s$,
- $I_{s,0} \cap I_{s,1} = \emptyset$, and
- $|I_s| < 1/(|s| + 1)$,

where $|I|$ denotes the length of interval I and $|s|$ denotes the length of sequence s.

In particular, if $\langle\rangle$ denotes the empty sequence, let $I_{\langle\rangle}$ be the closure of $I \cap P$ for some open interval I with length at most 1 whose intersection with P is nonempty. Then, given a set I_s satisfying the above properties, by Lemma 6.15 choose $I_{s,0}$ and $I_{s,1}$ to be disjoint closed subintervals of I_s shorter than $1/(|s|+2)$ whose intersection with P is nonempty.

Now, for all $f \in 2^\omega$, define

$$G(f) = \bigcap_{i \in \omega} I_{\overline{f}(i)},$$

where $\overline{f}(n) = \{f(0), f(1), \ldots, f(n)\}$. Actually, we are abusing notation a bit here: what we mean is that $G(f)$ is the unique member of the given intersection; we must show that this intersection does indeed result in a singleton set. This follows from the fact that we have an infinite intersection of nested, closed intervals of arbitrarily small length and that the real numbers are order-complete.

To see that $G(f) \in P$, note that $G(f)$ is an intersection of decreasing intervals, each of which has a nonempty intersection with P; if we pick one point from the intersection of each interval with P, they form a sequence with limit $G(f)$, which is contained in P since P is closed.

Finally, suppose $f, f' \in 2^\omega$ with $f \neq f'$. Let $n \in \omega$ be the smallest index for which $f(n) \neq f'(n)$. Then $I_{\overline{f}(n)} \cap I_{\overline{f'}(n)} = \emptyset$ by construction, and therefore $G(f) \cap G(f') = \emptyset$. This shows that G is injective. ☒

Theorem 6.17 (Cantor-Bendixson)**.** *If $C \subseteq \mathbb{R}$ is closed and uncountable, then there exists some perfect, nonempty $P \subseteq C$.*

Remark. In a sense, this is where set theory started. This proof is what motivated the development of transfinite ordinals, since it describes a recursive process that is not completed after the first limit stage.

Proof. Let $C \subseteq \mathbb{R}$ be closed. Define the *Cantor-Bendixson derivative*

$$C' = \{\, a \in C \mid a \text{ is a limit point of } C \,\}.$$

This operation maps closed sets to closed sets, since closed sets in $\mathbb{R}$ are those which contain all their limit points, and the derivative is monotone and retains all limit points. Then define

$$\begin{aligned} C_0 &= C \\ C_{\alpha+1} &= (C_\alpha)' \\ C_\lambda &= \bigcap_{\beta < \lambda} C_\beta \qquad (\lim(\lambda)). \end{aligned}$$

Note that C_β is closed for all β by induction.

Claim: $C_\gamma = C_{\gamma+1}$ for some γ. For if not, $C_\alpha \neq C_\beta$ for any $\alpha \neq \beta$, since C is monotone. Then $C_{(-)}$ would be an injection $Ord \to \mathcal{P}(C)$, which is absurd.

Note that C_γ is perfect, since it consists solely of limit points and is closed. If $C_\gamma \neq \emptyset$, we are done.

We claim that C_γ cannot be $\emptyset$ since this would imply that C is countable. Consider $C_\beta - C_{\beta+1}$, which contains all the isolated points in C_β. That is, if $a \in C_\beta - C_{\beta+1}$, there exists an open interval $I_a \ni a$ such that $C_\beta \cap I_a = \{a\}$. In particular, we may choose I_a to be an open interval with rational endpoints.

Note that each I_a is distinct; otherwise, at the earliest stage when I_a arose, it would have contained more than one point. Therefore, we have an injection from C into the set of intervals with rational endpoints, which is countable. ∎

Remark. The above proof shows that every closed set can be decomposed into a perfect subset and a countable subset. (In fact, it turns out that every closed set can be *uniquely* so decomposed.)

Definition 6.18. The smallest γ in the above proof for which $C_\gamma = C_{\gamma+1}$ is called the *Cantor-Bendixson rank* of C, and the above proof shows that $\gamma < \aleph_1$.

Exercise: construct closed sets whose Cantor-Bendixson rank is strictly greater than ω. In fact, it can be shown that for every $\gamma < \aleph_1$, there exists a closed $C \subseteq \mathbb{R}$ with Cantor-Bendixson rank γ.

Corollary 6.19. *For every $C \subseteq \mathbb{R}$, if C is closed and uncountable then $C \sim \mathbb{R}$. This follows from Lemma 6.16 and Theorem 6.17.*

Remark. We might hope that every uncountable set has a perfect subset; this, of course, would resolve the CH. However...

Theorem 6.20. *There exists a set X with* $\text{card}(X) = 2^{\aleph_0} = \text{card}(\mathbb{R} - X)$ *such that for every perfect set P, $P \not\subseteq X$ and $P \not\subseteq \mathbb{R} - X$.*

Proof. We use the AC to construct X. Let $P_\alpha, \alpha < 2^{\aleph_0}$ be an ordering of the perfect sets (there are $2^{\aleph_0}$ perfect sets; see Lemma 6.21). Also, let x_α be an ordering of $\mathbb{R}$. Now define r_γ to be the real number with next-to-least index in the sequence x_α which comes after all r_β, $\beta < \gamma$, and for which $r_\gamma \in P_\gamma$. We can keep picking such r_γ since each P_α has cardinality $2^{\aleph_0}$ and therefore cannot be contained in any initial segment of the x_α's. ∎

Lemma 6.21. *There are $2^{\aleph_0}$ perfect sets.*

Proof. There are at least $2^{\aleph_0}$ perfect sets, since there is an injection from $\mathcal{P}(\mathbb{N})$ to the set of all perfect sets: for each set of naturals, identify each natural with a small closed interal containing it, and take the union. There are at most $2^{\aleph_0}$ perfect sets since there are $2^{\aleph_0}$ closed sets (which, in turn, follows from the fact that any closed set can be expressed as a countable intersection of rational intervals). ∎