Lecture 9: Relative Consistency I

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7 The Axiom of Regularity

Remark. We will now move more towards logic. We want to be able to show various independence results, such as that the consistency of ZF implies the consistency of ZF + AC + CH (and also ZF + AC + ¬CH). In some sense we can think of this as the "metamathematics" of set theory.

Definition 7.1 (Axiom of Regularity (Reg))**.** Every set has an $\in$-minimal element. Formally:

$$\forall x.(\exists y.y \in x) \implies \exists y.(y \in x \wedge (\forall z.z \in y \implies z \notin x)).$$

Remark. This axiom implies that we cannot have a set x which is an element of itself; then the set $\{x\}$ does not satisfy the axiom. We also cannot have a cyclic chain of inclusions $x_1 \in x_2 \in x_3 \in \cdots \in x_1$, or an infinite descending chain $x_1 \ni x_2 \ni x_3 \ni \ldots$; in either case, the set $\{x_1, x_2, x_3, \ldots\}$ fails to satisfy the axiom of regularity.

Note that we did not particularly need this axiom for the theory of Ord, $\mathbb{Q}$, $\mathbb{R}$, and so on, since all of those classes are well-founded by definition. But it will become convenient to restrict ourselves to well-founded sets when talking about models of set theory.

One question to ask ourselves is, given Reg, could we still have non-well-founded classes? The answer, it turns out, is no.

Definition 7.2. The *transitive closure* $TC(x)$ of a set x is defined as follows:

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ TC(x) &= \bigcup_{n \in \omega} x_n. \end{aligned}$$

Lemma 7.3. *$TC(x)$ is the $\subseteq$-least transitive set y such that $x \subseteq y$.*

Proof. First we show that $TC(x)$ is transitive. Suppose $y \in TC(x)$, and $z \in y$. By definition, $y \in x_n$ for some $n \in \omega$. But then $z \in x_{n+1}$; therefore, $TC(x)$ is transitive.

Now, if $x \subseteq y$ and y is transitive, we will show that $TC(x) \subseteq y$. It suffices to show that $x_n \subseteq y$ for all n, which we show by induction. The base case holds by assumption. For the inductive case, suppose $x_m \subseteq y$. Then if $z \in x_{m+1}$, by definition, $z \in z' \in x_m$ for some z'; but then, since $x_m \subseteq y$ and y is transitive, $z \in y$. ∎

Lemma 7.4 (Regularity for classes)**.**

$$\exists x.\varphi(x) \implies \exists x.(\varphi(x) \wedge \forall y.(\varphi(y) \implies y \notin x))$$

Proof. Suppose φ is some predicate, and that $\varphi(u)$ holds. Then define

$$z = \{ x \mid x \in TC(u) \wedge \varphi(x) \}.$$

If z is empty, then u is $\in$-minimal for φ. Otherwise, note that z is a set by comprehension, so by Reg, it has an $\in$-minimal element, call it y. Then y is $\in$-minimal for φ. For if $q \in y$ and $\varphi(q)$, then $q \in TC(u)$ by transitivity of $TC(u)$, and hence $q \in z$. But this contradicts the minimality of y. ∎

Definition 7.5. Recall the definition of the transfinite hierarchy of sets, V:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha. \end{aligned}$$

We write $V(x)$ if and only if there is some α for which $x \in V_\alpha$.

Definition 7.6 (Rank)**.** The *rank* of a set x, denoted $\operatorname{rank}(x)$, is the least α for which $x \in V_{\alpha+1}$.

Theorem 7.7. *Under ZFC, the transfinite hierarchy of sets contains all sets. Formally,*

$$\text{ZFC} \vdash \forall x.V(x).$$

Proof. Suppose there is some set x for which $\neg V(x)$. Let u be the $\in$-minimal such set (by Regularity). Then for every $w \in u$, there is some α for which $w \in V_\alpha$. Therefore, rank is a functional relation on u. Now consider $\sup(\operatorname{rank}[u]) = \beta$; we claim that $u \subseteq V_{\beta+1}$. Consider $x \in u$; by definition, $\beta \geq \operatorname{rank}(x)$, so $x \in V_{\beta+1}$, since the V_α are cumulative. Therefore $u \subseteq V_{\beta+1}$, a contradiction. ∎

Remark. This shows that every class which is bounded in rank is a set. Conversely, every class which is not bounded in rank is not a set.

We will now start in on proving some relative consistency results.

Theorem 7.8. *If ZF without Regularity is consistent, then so is ZF.*

Remark. We will show this by proving that from ZF without Regularity, we can prove the "relativization" of the ZF axioms to V.

Definition 7.9. The *relativization* of a formula φ to V, denoted φ^V, is defined as follows. All atomic formulas ($\in$, $=$) translate to themselves. $(-)^V$ commutes past $\wedge$, $\vee$, and $\neg$. The only interesting cases are $\forall$ and $\exists$:

$$\begin{aligned} [\exists x.\varphi]^V &= \exists x.V(x) \wedge \varphi^V \\ [\forall x.\varphi]^V &= \forall x.V(x) \implies \varphi^V \end{aligned}$$

That is, we change quantifiers into "bounded quantifiers" which have a universe of V.

(ZF)^V indicates the set of axioms of ZF, each relativized to V.

Definition 7.10. Δ_0 is the smallest set of formulas containing atoms ($x \in y$ or $x = y$) and closed under connectives and bounded quantifiers (*e.g.* $\forall x \in z.\varphi$).

Definition 7.11. φ is *absolute* for M iff for all $\overline{x} \in M$,

$$\varphi^M(\overline{x}) \iff \varphi(\overline{x}).$$

Remark. As usual, we take $\overline{x} \in M$ to indicate a sequence of elements of M.

Lemma 7.12. *If M is transitive and φ is Δ_0, then φ is absolute for M.*

Proof. By structural induction on φ. First, if φ is an atom, then $\varphi^M(\overline{x}) = \varphi(\overline{x})$. If the top-level constructor of φ is $\wedge$, $\vee$, or $\neg$, the result follows immediately by definition of relativization and the induction hypothesis.

Now suppose φ is of the form $\exists x \in a.\varphi'$, that is, $\exists x.x \in a \wedge \varphi'$. By the induction hypothesis, we know that φ' is absolute for M. Note that

$$\varphi^M = \exists x.M(x) \wedge x \in a \wedge \varphi'^M.$$

$\big[\varphi^M(\overline{x}) \implies \varphi(\overline{x}).\big]$ Let $\overline{x} \in M$, and suppose $\varphi^M(\overline{x})$; we wish to show that $\varphi(\overline{x})$. Let y be the set that witnesses $\varphi^M(\overline{x})$. Then we can show that y also witnesses $\varphi(\overline{x})$. We know that $y \in a$ from $\varphi^M(\overline{x})$. However, φ' may contain x free; we must show $\varphi'(\overline{x}, y)$. This follows from the induction hypothesis if $y \in M$: but $\varphi^M(\overline{x})$ gives us $M(y)$.

$\big[\varphi(\overline{x}) \implies \varphi^M(\overline{x}).\big]$ Now suppose $\varphi(\overline{x})$, and let y be the witness. y is also a witness of $\varphi^M(\overline{x})$; the argument is similar, except we also need to show that $M(y)$ holds. We know that $y \in a$; but a is free in φ, so in $\varphi(\overline{x})$ it has been replaced by some element of $\overline{x}$, which is in M by assumption. M is transitive, so this implies that $y \in M$ as well.

Finally, suppose φ is of the form $\forall x \in a.\varphi'$, that is, $\forall x.x \in a \Rightarrow \varphi'$. Then we have

$$\varphi^M = \forall x.M(x) \Rightarrow x \in a \Rightarrow \varphi'^M.$$

$\big[\varphi^M(\overline{x}) \implies \varphi(\overline{x}).\big]$ Omitted. (For now. Maybe.) ∎