Lecture 10: A Digression on Absoluteness

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8 A digression

Remark. An example of a formula which is *not* Δ_0 is the formula $\varphi(x)$ which states $\mathrm{card}(x) = \omega$, that is,

$$\exists f.(f : \omega \xrightarrow[\text{onto}]{\text{1-1}} x). \tag{1}$$

Note that $\exists f$ is an *unbounded* quantifier.

Moreover, it is the case that φ is not absolute for transitive universes, demonstrating (by Lemma 7.12) that it is not possible to find *any* Δ_0 formula expressing the same property. We will spend the rest of the lecture exploring why.

Definition 8.1. B is an *elementary substructure* (or *elementary submodel*) of A, denoted $B \preceq A$, iff for all formulas $\varphi(\overline{x})$ and $\overline{b} \in B$,

$$B \models \varphi[\overline{b}] \iff A \models \varphi[\overline{b}].$$

Definition 8.2. A and B are *elementarily equivalent*, denoted $A \equiv B$, iff for all φ,

$$A \models \varphi \iff B \models \varphi.$$

Remark. For example, consider the structures $A = \langle \omega, < \rangle$ and $B = \langle \omega - \{0\}, < \rangle$. These are isomorphic, and therefore $A \equiv B$. However, it is not the case that $B \preceq A$: for example, if $\varphi(x)$ denotes "x has no predecessor," then $B \models \varphi(1)$ but $A \not\models \varphi(1)$. (Also, $A \not\preceq B$ since A is not a subset of B.)

Lemma 8.3 (Mostowski's Collapsing Lemma)**.** *If $A = \left\langle A, E^A \right\rangle$ is a well-founded extensional model of ZF, then A is isomorphic to a transitive set.*

Proof. Suppose $\left\langle A, E^A \right\rangle$ is a well-founded, extensional model of ZF. Then define $f : A \to V$ by

$$f(a) = \{\, f(b) \mid E^A(b, a) \,\}.$$

(Note that we may recursively define f in this way since $\left\langle A, E^A \right\rangle$ is well-founded.) Then we must show that $f[A]$ is transitive, and that f is an isomorphism.

First, we show that $f[A]$ is transitive. Let $x \in y \in f[A]$. Then $y = f(a)$ for some $a \in A$, and $y = \{\, f(b) \mid E^A(b, a) \,\}$. Therefore, $x = f(b)$ for some b with $E^A(b, a)$, which means that $x \in f[A]$.

Now, we must show f is an isomorphism between A and $f[A]$. Clearly it is surjective, so we need only show it is structure-preserving. Suppose $E^A(b, a)$; then $f(b) \in f(a)$ by definition of $f(a)$. ∎

Remark. We interrupt this lecture to bring you the following digression within a digression.

> *Remark.* In the statement of Lemma 8.3, why do we need to state that A is a *well-founded* model of ZF? Doesn't this follow from the axiom of regularity and the fact that it is a model?
>
> The somewhat surprising answer is: no! "Just because A *thinks* it is well-founded..." In fact, we can actually show the following theorem.
>
> **Theorem 8.4.** *If A is an infinite structure with arbitrarily long finite chains, then there exists a non-well-founded structure B such that $B \equiv A$.*
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> To prove this theorem, we first need a few more tools.
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> **Theorem 8.5** (Compactness of first-order logic (Gödel)). *For any set of first-order sentences T, if every finite $S \subseteq T$ is satisfiable, then T is satisfiable.*
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> *Remark.* Gödel first showed this as a corollary to his completeness theorem for first-order logic.
>
> **Theorem 8.6** (Completeness of first-order logic (Gödel)). *For every formula φ of first-order logic, if $T \models \varphi$, then $T \vdash \varphi$.*
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> *Proof.* We prove that Theorem 8.5 is a corollary to Theorem 8.6, by showing the contrapositive. Suppose that T is not satisfiable. Then $T \models \varphi \wedge \neg\varphi$, vacuously; so, by Theorem 8.6, $T \vdash \varphi \wedge \neg\varphi$. Proofs must be finite, so the proof must use only a finite set S of formulas in T. Hence $S \vdash \varphi \wedge \neg\varphi$, and by the soundness of first-order logic, $S \models \varphi \wedge \neg\varphi$. Therefore S is not satisfiable. ∎
>
> *Remark.* There are actually at least three other ways to show Theorem 8.5. One was shown by some guy using structures with constants, or something like that. One was shown by some other guy using ultraproducts, whatever those are (we might see this later in the course). Finally, there are topological methods involving scary things named for people.
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> *Proof.* We are now in a position to prove Theorem 8.4. Suppose A is a structure with arbitrarily long chains. Now define
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> $$T = Th(A) \cup \{ E(c_{n+1}, c_n) \mid n \in \omega \},$$
>
> where E is the relation of A, the c_i are new constant symbols, and $Th(A) = \{ \varphi \mid A \models \varphi \}$. Since A contains arbitrarily long finite chains, any finite subset of T is satisfiable by assigning appropriate elements of A to the c_i. By Theorem 8.5, T is satisfiable, that is, there exists a structure B such that $B \models T$. Clearly, B cannot be well-founded, because it contains an

infinite decreasing chain. Note that $A \models \varphi \implies B \models \varphi$ by construction. The converse is also true, which we can show by contradiction: if $B \models \varphi$ but $A \not\models \varphi$, then $A \models \neg\varphi$, implying that $B \models \neg\varphi$, a contradiction. Thus, $A \equiv B$. ∎

Remark. This means that ZF—even with the Axiom of Regularity—has non-well-founded models! Note, however, that any infinitely descending chain in such a model is *not* represented by an element in the universe.

Consider also the naturals with addition, multiplication, successor, and zero, ordered by $<$, which is a well-founded relation. The preceding theorem shows that there are models of these axioms which are not well-founded! Such a model has "non-standard naturals"; each of these have successors and predecessors which are also non-standard, so each "sprouts" a "$\mathbb{Z}$-chain". Similarly, each of the elements in a $\mathbb{Z}$-chain is $a+n$ for some standard n, $a+a$ must sprout a different $\mathbb{Z}$-chain, and so on...

Remark. And now, back to your regularly scheduled digression.

Theorem 8.7 (Löwenheim-Skolem)**.** *If $A = \left\langle A, E^A \right\rangle$ is a structure (that is, a set with a binary relation) such that $A \models T$, then for all countable $X \subseteq A$, there is some countably infinite B with $X \subseteq B \subseteq A$, and $B \preceq A$.*

Proof. (Deferred to a later lecture.) ∎

Remark. With the machinery we have now developed, we can show that equation (1) is not absolute for all transitive universes—that is, it could be true in some universe, but *not* true in some relativization of that universe which is still a model.

Theorem 8.8. *There is some transitive universe M for which*

$$\varphi(x) = \exists f.(f : \omega \xrightarrow[\textit{onto}]{\textit{1-1}} x)$$

is not absolute.

Proof. Let $\langle A, E\rangle$ be a well-founded, extensional model of ZF. By Löwenheim-Skolem, we can find a countable model of ZF, B, which is an elementary submodel of A (and hence well-founded and extensional). Then, by Mostowski, B is isomorphic to a transitive set C. Since C is a model of ZF, it must contain some element x satisfying the formula "$x = \mathcal{P}(\omega)$", and it must be the case that

$$C \models \neg\exists f.(f : \omega \xrightarrow[\text{onto}]{\text{1-1}} x)$$

(this is just Cantor's Theorem). But C is countable, and since C is transitive, x must be countable also. Hence, "countableness" is not an absolute property. ∎

Remark. This is known as Skolem's Paradox, and gives additional insight into the limits of first-order logic to express properties of sets.