9 Relative consistency of Reg

We now finally return to prove Theorem 7.8:

**Theorem 7.8.** If ZF without Regularity is consistent, then so is ZF.

We'll first need a few more lemmas.

**Lemma 9.1.** The rank hierarchy $V$ (Definition 7.5) is transitive.

*Proof.* By definition of $V$, it suffices to show that $V_\alpha$ is transitive for all ordinals $\alpha$, which we show by transfinite induction.

- $V_0 = \emptyset$, which is vacuously transitive.
- By definition, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$; by the inductive hypothesis we may assume $V_\alpha$ is transitive. Let $x \subseteq V_\alpha$. Now let $y \in x$; then $y \in V_\alpha$. But since $V_\alpha$ is transitive, this means that $y \subseteq V_\alpha$, and hence $y \in V_{\alpha+1}$.
- Now consider $V_\lambda = \bigcup_{\beta < \lambda} V_\beta$, where $\lambda$ is a limit ordinal. Let $x \in V_\lambda$. Then $x \in V_\beta$ for some $\beta < \lambda$. Since $V_\beta$ is transitive by the inductive hypothesis, if $y \in x$, then $y \in V_\beta$, and hence $y \in V_\lambda$.

Remark. This immediately implies that the rank hierarchy is cumulative: $V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$, and since $V_{\alpha+1}$ is transitive, $V_\alpha \subseteq V_{\alpha+1}$ as well.

**Lemma 9.2.** If all the elements of a set $u$ are sets in the rank hierarchy, then so is $u$.

*Proof.* Let $\alpha$ be the maximum rank of the elements of $u$; since the rank hierarchy is cumulative, $u \subseteq V_\alpha$. But then $u \in V_{\alpha+1}$.

**Lemma 9.3.** If $x$ is in the rank hierarchy, so is $\mathcal{P}(x)$.

*Proof.* Suppose $x \in V$. Therefore $x \subseteq V$, since $V$ is transitive. Then by the previous lemma, every subset of $x$ is in $V$. Applying the previous lemma again, we conclude that $\mathcal{P}(x) \in V$.

*Proof of Theorem 7.8.* We must show that if we assume ZF - Reg, each of the axioms of ZF holds when relativized to $V$.

- Axiom of Extensionality. We must show
  \[
  [\forall x. \forall y. ((\forall z. z \in y) \land (\forall z. z \in x)) \Rightarrow (x = y)]^V.
  \]

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By definition of \((-)^V\), this is equivalent to

$$\forall x \in V. \forall y \in V.[((\forall z \in x. z \in y) \land (\forall z \in y. z \in x)) \Rightarrow (x = y)]^V.$$  

(where \(\forall x \in V. \varphi\) is an abbreviation for \(\forall x. V(x) \Rightarrow \varphi\)). Let \(x \in V\) and \(y \in V\); then we must show the remainder of the formula for this particular \(x\) and \(y\). However, note that this formula is \(\Delta_0\) and its free variables are in \(V\) (which is transitive), so by Lemma 7.12, its relativization holds iff the unrelativized version holds in the original universe—which is does, by the Axiom of Extensionality.

- **Pairing.** We must show

  $$[\forall x. \forall y. \exists z. \forall w. (w \in z \iff (w = x \lor w = y))]^V,$$

  that is,

  $$\forall x \in V. \forall y \in V. \exists z \in V. \forall w \in V. (w \in z \iff (w = x \lor w = y)).$$

  So, let \(x, y \in V\), and let \(z = \{x, y\}\), which is guaranteed to exist by the Axiom of Pairing. Note that \(z \in V\), since its elements are (by Lemma 9.2). The remaining condition holds for all sets \(w\) by the Axiom of Pairing, so it certainly holds for all sets \(w \in V\).

- **Union.** We must show

  $$[\forall x. \exists y. \forall z. (z \in y \iff (\exists w \in x. z \in w))]^V,$$

  that is,

  $$\forall x \in V. \exists y \in V. \forall z \in V. [z \in y \iff (\exists w \in x. z \in w)]^V,$$

  noting that the part still in brackets is \(\Delta_0\). Let \(x \in V\), and let \(y = \bigcup x\) (which exists by the Axiom of Union). \(y \in V\), again by Lemma 9.2. Finally, the formula in brackets holds for all sets \(z\), so the relativized version certainly holds for all \(z \in V\), since it is \(\Delta_0\).

- **Power set.** We must show

  $$[\forall x. \exists y. \forall z. (z \in y \iff (\forall w \in z. w \in x))]^V,$$

  that is,

  $$\forall x \in V. \exists y \in V. \forall z \in V. [z \in y \iff (\forall w \in z. w \in x)]^V.$$

  Let \(x \in V\), and let \(y = \mathcal{P}(x)\). By Lemma 9.3, \(y \in V\). The remainder of the argument is similar to the previous case.
• Infinity. We must show

\[ \exists x.\emptyset \in x \land (\forall y \in x.y \cup \{ y \} \in x) \] 

that is,

\[ \exists x \in V.\left( \emptyset \in x \land (\forall y \in x.y \cup \{ y \} \in x) \right) \]

Note that the formula inside the brackets can be expressed as a \( \Delta_0 \) formula. Let \( x = \omega \), and note that it satisfies the Axiom of Infinity, and is in \( V \) (in particular, it is in \( V_{\omega+1} \)). Then we are done, since the remainder of the formula is \( \Delta_0 \).

• Regularity. We must show

\[ \forall x. (\exists y \in x) \Rightarrow \exists y \in x.\forall z. z \not\in x \]

(without using the Axiom of Regularity!). All but the \( \forall x. \) is clearly \( \Delta_0 \). So let \( x \in V \), and suppose \( x \) is not empty. Pick \( y \) of minimal rank in \( x \). Then \( y \cap x = \emptyset \), since otherwise there would be some element of \( x \) which is also an element of \( y \), contradicting the minimality of the rank of \( y \).

• Separation. We must show that for all formulas \( \varphi \),

\[ \forall t. \forall x. \exists y. \forall z. z \in y \Leftrightarrow z \in x \land \varphi(z, t) \]

that is,

\[ \forall t \in V. \forall x \in V. \exists y \in V. z \in y \Leftrightarrow z \in x \land \varphi^V(z, t) \]

So, let \( t, x \in V \). Then let \( y = \{ z \in x \mid \varphi^V(z, t) \} \), which exists by the Axiom of Separation. But all the elements of \( y \) are elements of \( x \in V \), and therefore also elements of \( V \) since \( V \) is transitive; but then by Lemma 9.2, \( y \in V \).

• Replacement. We must show that for all \( F \),

\( ( F \text{ is a functional relation})^V \Rightarrow (\forall x. \exists y. y = F[x])^V \),

that is, more explicitly,

\( (\forall x. (\exists y. F(x, y) \land (\forall y, y'. F(x, y) \land F(x, y') \Rightarrow y = y')))^V \Rightarrow (\forall x. \exists y. y = F[x])^V \).

So, we are given the fact that \( F^V \) is a functional relation when restricted to \( V \), and that it sends every element of \( V \) to another element of \( V \). Let \( x \in V \). Now, invoking the Axiom of Replacement, we may conclude that the image of \( x \) under \( F^V \restriction V \) is a set. However, since all the elements of \( x \) are elements of \( V \) (since \( V \) is transitive), this image is a set of elements of \( V \), and hence in \( V \). Furthermore, this image \( y \) should satisfy

\( (y = F[x])^V \),

that is,

\( (\forall z. (z \in y \Leftrightarrow \exists w \in x. F(w, z)))^V \),

but this is clearly satisfied by the image of \( x \) under \( F^V \restriction V \).
• Choice. We must show that

\[ \forall x. (\forall y \in x. y \neq \emptyset) \Rightarrow \exists f. \text{dom}(f) = x \land \forall y \in x. f(y) \in y \] ^V ,

that is,

\[ \forall x \in V. (\forall y \in x. y \neq \emptyset) \Rightarrow \exists f \in V. \text{dom}(f) = x \land \forall y \in x. f(y) \in y. \]

So, suppose \( x \in V \) and all the elements of \( x \) are nonempty. Then by the Axiom of Choice, there exists a choice function \( f \) in the universe which clearly satisfies the necessary conditions on \( f \). Also, \( f \) consists of pairs of elements of \( x \) and elements of elements of \( x \), all of which are in \( V \) by transitivity; since \( V \) contains pairs by construction, \( f \in V \). \qed