Lecture 12: Relative Consistency II

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9 Relative consistency of Reg

We now finally return to prove Theorem 7.8:

Theorem 7.8. *If ZF without Regularity is consistent, then so is ZF.*

We'll first need a few more lemmas.

Lemma 9.1. *The rank hierarchy V (Definition 7.5) is transitive.*

Proof. By definition of V, it suffices to show that V_α is transitive for all ordinals α, which we show by transfinite induction.

- $V_0 = \emptyset$, which is vacuously transitive.
- By definition, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$; by the inductive hypothesis we may assume V_α is transitive. Let $x \in V_{\alpha+1}$. Then $x \subseteq V_\alpha$. Now let $y \in x$; then $y \in V_\alpha$. But since V_α is transitive, this means that $y \subseteq V_\alpha$, and hence $y \in V_{\alpha+1}$.
- Now consider $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$, where λ is a limit ordinal. Let $x \in V_\lambda$. Then $x \in V_\beta$ for some $\beta < \lambda$. Since V_β is transitive by the inductive hypothesis, if $y \in x$, then $y \in V_\beta$, and hence $y \in V_\lambda$. ∎

Remark. This immediately implies that the rank hierarchy is cumulative: $V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$, and since $V_{\alpha+1}$ is transitive, $V_\alpha \subseteq V_{\alpha+1}$ as well.

Lemma 9.2. *If all the elements of a set u are sets in the rank hierarchy, then so is u.*

Proof. Let α be the maximum rank of the elements of u; since the rank hierarchy is cumulative, $u \subseteq V_\alpha$. But then $u \in V_{\alpha+1}$. ∎

Lemma 9.3. *If x is in the rank hierarchy, so is $\mathcal{P}(x)$.*

Proof. Suppose $x \in V$. Therefore $x \subseteq V$, since V is transitive. Then by the previous lemma, every subset of x is in V. Applying the previous lemma again, we conclude that $\mathcal{P}(x) \in V$. ∎

Proof of Theorem 7.8. We must show that if we assume ZF - Reg, each of the axioms of ZF holds when relativized to V.

- Axiom of Extensionality. We must show

$$\left[\forall x.\forall y.((\forall z \in x.z \in y) \wedge (\forall z \in y.z \in x)) \Rightarrow (x = y)\right]^V.$$

By definition of $(-)^V$, this is equivalent to

$$\forall x \in V. \forall y \in V. \big[((\forall z \in x. z \in y) \wedge (\forall z \in y. z \in x)) \Rightarrow (x = y)\big]^V.$$

(where $\forall x \in V.\varphi$ is an abbreviation for $\forall x.V(x) \Rightarrow \varphi$). Let $x \in V$ and $y \in V$; then we must show the remainder of the formula for this particular x and y. However, note that this formula is Δ_0 and its free variables are in V (which is transitive), so by Lemma 7.12, its relativization holds iff the unrelativized version holds in the original universe—which is does, by the Axiom of Extensionality.

- Pairing. We must show

$$\big[\forall x. \forall y. \exists z. \forall w. (w \in z \Leftrightarrow (w = x \vee w = y))\big]^V,$$

that is,

$$\forall x \in V. \forall y \in V. \exists z \in V. \forall w \in V. (w \in z \Leftrightarrow (w = x \vee w = y)).$$

So, let $x, y \in V$, and let $z = \{x, y\}$, which is guaranteed to exist by the Axiom of Pairing. Note that $z \in V$, since its elements are (by Lemma 9.2). The remaining condition holds for all sets w by the Axiom of Pairing, so it certainly holds for all sets $w \in V$.

- Union. We must show

$$\big[\forall x. \exists y. \forall z. z \in y \Leftrightarrow (\exists w \in x. z \in w)\big]^V,$$

that is,

$$\forall x \in V. \exists y \in V. \forall z \in V. \big[z \in y \Leftrightarrow (\exists w \in x. z \in w)\big]^V,$$

noting that the part still in brackets is Δ_0. Let $x \in V$, and let $y = \bigcup x$ (which exists by the Axiom of Union). $y \in V$, again by Lemma 9.2. Finally, the formula in brackets holds for all sets z, so the relativized version certainly holds for all $z \in V$, since it is Δ_0.

- Power set. We must show

$$\big[\forall x. \exists y. \forall z. z \in y \Leftrightarrow (\forall w \in z. w \in x)\big]^V,$$

that is,

$$\forall x \in V. \exists y \in V. \forall z \in V. \big[z \in y \Leftrightarrow (\forall w \in z. w \in x)\big]^V.$$

Let $x \in V$, and let $y = \mathcal{P}(x)$. By Lemma 9.3, $y \in V$. The remainder of the argument is similar to the previous case.

- Infinity. We must show

$$\left[\exists x.\emptyset \in x \wedge (\forall y \in x.y \cup \{y\} \in x)\right]^V,$$

that is,

$$\exists x \in V.\left[\emptyset \in x \wedge (\forall y \in x.y \cup \{y\} \in x)\right]^V.$$

Note that the formula inside the brackets can be expressed as a Δ_0 formula. Let $x = \omega$, and note that it satisfies the Axiom of Infinity, and is in V (in particular, it is in $V_{\omega+1}$). Then we are done, since the remainder of the formula is Δ_0.

- Regularity. We must show

$$\left[\forall x.(\exists y \in x) \Rightarrow \exists y \in x.\forall z \in y.z \notin x\right]^V$$

(without using the Axiom of Regularity!). All but the $\forall x.$ is clearly Δ_0. So let $x \in V$, and suppose x is not empty. Pick y of minimal rank in x. Then $y \cap x = \emptyset$, since otherwise there would be some element of x which is also an element of y, contradicting the minimality of the rank of y.

- Separation. We must show that for all formulas φ,

$$\left[\forall \bar{t}.\forall x.\exists y.\forall z.z \in y \Leftrightarrow z \in x \wedge \varphi(z, \bar{t})\right]^V,$$

that is,

$$\forall \bar{t} \in V.\forall x \in V.\exists y \in V.z \in y \Leftrightarrow z \in x \wedge \varphi^V(z, \bar{t}).$$

So, let $\bar{t}, x \in V$. Then let $y = \{\, z \in x \mid \varphi^V(z, \bar{t}) \,\}$, which exists by the Axiom of Separation. But all the elements of y are elements of $x \in V$, and therefore also elements of V since V is transitive; but then by Lemma 9.2, $y \in V$.

- Replacement. We must show that for all F,

$$(F \text{ is a functional relation})^V \Rightarrow (\forall x.\exists y.y = F[x])^V,$$

that is, more explicitly,

$$(\forall x.(\exists y.F(x,y) \wedge (\forall y, y'.F(x,y) \wedge F(x,y') \Rightarrow y = y')))^V \Rightarrow (\forall x.\exists y.y = F[x])^V.$$

So, we are given the fact that F^V is a functional relation when restricted to V, and that it sends every element of V to another element of V. Let $x \in V$. Now, invoking the Axiom of Replacement, we may conclude that the image of x under $F^V \restriction V$ is a set. However, since all the elements of x are elements of V (since V is transitive), this image is a set of elements of V, and hence in V. Furthermore, this image y should satisfy

$$(y = F[x])^V,$$

that is,

$$(\forall z.(z \in y \Leftrightarrow \exists w \in x.F(w, z)))^V,$$

but this is clearly satisfied by the image of x under $F^V \restriction V$.

- Choice. We must show that

  $$\left[\forall x.(\forall y \in x.y \neq \emptyset) \Rightarrow \exists f.\,\mathrm{dom}(f) = x \wedge \forall y \in x.f(y) \in y\right]^V,$$

  that is,

  $$\forall x \in V.(\forall y \in x.y \neq \emptyset) \Rightarrow \exists f \in V.\,\mathrm{dom}(f) = x \wedge \forall y \in x.f(y) \in y.$$

  So, suppose $x \in V$ and all the elements of x are nonempty. Then by the Axiom of Choice, there exists a choice function f in the universe which clearly satisfies the necessary conditions on f. Also, f consists of pairs of elements of x and elements of elements of x, all of which are in V by transitivity; since V contains pairs by construction, $f \in V$. SDG