Lecture 15: The Reflection Principle

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12 ZF is not finitely axiomatizable

Definition 12.1. A sequence of sets indexed by ordinals, M_α, is a *cumulative hierarchy* iff

1. $M_\alpha \subseteq M_{\alpha+1} \subseteq \mathcal{P}(M_\alpha)$, for all α, and
2. $\bigcup_{\alpha<\lambda} M_\alpha = M_\lambda$, for $\lim(\lambda)$.

Remark. For example, V (the rank hierarchy) is a cumulative hierarchy, as is L (the constructible hierarchy, to be covered later). Generally, if the sequence M_α is a cumulative hierarchy, we write $M(x)$ to denote the predicate $\exists\alpha.x \in M_\alpha$.

Definition 12.2. A class of ordinals C is closed unbounded (abbreviated *club*) iff

- it is *closed*, that is, $C(\lambda)$ holds for limit ordinals λ whenever, for every $\beta < \lambda$, there is some $\beta < \gamma < \lambda$ with $C(\gamma)$; and
- it is *unbounded*, that is, for every ordinal α there exists some ordinal $\beta > \alpha$ with $C(\beta)$.

Remark. A set of ordinals is club iff it is the image of a normal function.

Lemma 12.3. *If C and D are closed unbounded, then so is $C \cap D$.*

Proof. We must show that $C \cap D$ is closed, and unbounded.

- $C \cap D$ is closed. Let λ be a limit ordinal, and suppose that for every $\beta < \lambda$ there is some $\beta < \gamma < \lambda$ with $C(\gamma)$ and $D(\gamma)$. Then $\lambda \in C$ and $\lambda \in D$, hence $\lambda \in C \cap D$.
- $C \cap D$ is unbounded. Let β be an ordinal, and define a sequence $\langle\alpha_i\rangle$ such that $\beta < \alpha_0 < \alpha_1 < \dots$ and $\alpha_{2i} \in C$, $\alpha_{2i+1} \in D$. We can construct such a sequence since C and D are unbounded. The sup of this sequence is larger than β, and in both C and D. ∎

Lemma 12.4. *For any map F which sends ordinals to ordinals, the class*

$$C = \{\, \alpha \mid \forall\beta.\beta < \alpha \Rightarrow F(\beta) < \alpha \,\}$$

is closed unbounded.

Remark. This seems quite magical! It is not even obvious that C should be nonempty. In some sense it asserts that infinitely many "strong limits" exist with respect to any map F, not just $\alpha \mapsto 2^\alpha$.

Proof. We must show that C is closed, and that it is unbounded.

- C is closed. Suppose $\lim(\lambda)$ and there is some increasing sequence ξ below λ contained in C. Pick any $\beta < \lambda$. Then since ξ is increasing, and λ is a limit ordinal, there must be some $\beta < \alpha < \lambda$ with $\alpha \in \xi$, that is, $\alpha \in C$. But then $F(\beta) < \alpha < \lambda$, so $\lambda \in C$.

- C is unbounded. Suppose γ is an ordinal; we wish to show there is some $\delta > \gamma$ with $\delta \in C$.

  Define

$$\begin{aligned} \gamma_0 &= \gamma \\ \gamma_{n+1} &= 1 + \sup_{\alpha<\gamma_n} \{F(\alpha)\} \\ \delta &= \sup_{n\in\omega} \{\gamma_n\}. \end{aligned}$$

  Now pick $\beta < \delta$; we wish to show that $F(\beta) < \delta$, from which it will follow that $\delta \in C$. By definition of δ, there is some n for which $\beta < \gamma_n$. But then $F(\beta) \leq \sup_{\alpha<\gamma_n}\{F(\alpha)\} < \gamma_{n+1} \leq \delta$. ⊠

Theorem 12.5 (Reflection principle)**.** *For every cumulative hierarchy M and formula $\varphi(x_1, \ldots, x_n)$, there is a closed unbounded class C of ordinals such that for every $\alpha \in C$,*

$$\forall \overline{x} \in M_\alpha . \varphi^{M_\alpha}(\overline{x}) \Leftrightarrow \varphi^M(\overline{x}).$$

Remark. If $M = V$, $\varphi^M = \varphi^V = \varphi$ under Regularity; hence every formula φ is reflected by some closed unbounded class of ranks.

Definition 12.6. A theory T is *reflexive* if $T \vdash Con(\varphi)$ for every $\varphi \in Conseq(T)$ (where Con denotes "is consistent" and $Conseq(T)$ denotes the set of all formulas derivable in T).

Remark. By Gödel's second incompleteness theorem, a reflexive theory (which is strong enough for the theorem to apply) can't be finitely axiomatizable. If it were, there would be some formula (the conjunction of the axioms) from which the entire theory would follow; but since the theory is reflexive it would then be able to prove its own consistency.

Moreover, the reflection principle implies that ZF is reflexive, and hence is not finitely axiomatizable. However, we will later give a more set-theoretic proof of this using the reflection principle, without appealing to Gödel.

Proof of Theorem 12.5. By induction on φ. (We may assume that $\forall$ and $\vee$ are encoded in terms of $\exists$, $\neg$, and $\wedge$.)

- If φ is an atomic formula ($x_1 \in x_2$ or $x_1 = x_2$) we may take C to be the class of all ordinals. (Relativization is the identity on atomic formulas.)

- $\varphi = \neg\theta$. By the inductive hypothesis, there is a club class C_θ corresponding to θ; we may take $C_\varphi = C_\theta$, since the condition for φ is equivalent to the condition for θ.

- $\varphi = \theta \wedge \psi$. If C_θ and C_ψ are the club classes from the inductive hypotheses, then $C_\varphi = C_\theta \cap C_\psi$ (which is club by Lemma 12.3) reflects φ, since if $\theta^{M_\alpha} \Leftrightarrow \theta^M$ and $\psi^{M_\alpha} \Leftrightarrow \psi^M$ both hold, then so does $\theta^{M_\alpha} \wedge \psi^{M_\alpha} \Leftrightarrow \theta^M \wedge \psi^M$, which is equivalent to $(\theta \wedge \psi)^{M_\alpha} \Leftrightarrow (\theta \wedge \psi)^M$.

- $\varphi = \exists y.\zeta(\overline{x}, y)$; let C_ζ reflect $\zeta(\overline{x}, y)$ by the inductive hypothesis.

  Now define $G(\overline{x})$ to be the least α such that there is some $y \in M_\alpha$ with $\zeta^M(\overline{x}, y)$, or 0 if there is no such α. In other words, for a given $\overline{x}$, $G(\overline{x})$ is the smallest rank that reflects φ for that particular $\overline{x}$.

  Furthermore, define

  $$F(\beta) = \sup\{\, G(\overline{x}) \mid \overline{x} \in M_\beta \,\}.$$

  Now, we claim that $C_\varphi = C_\zeta \cap \{\, \alpha \mid \lim(\alpha) \,\} \cap \{\, \alpha \mid \forall\beta.\beta < \alpha \Rightarrow F(\beta) < \alpha \,\}$ satisfies the requirements of the reflection principle. Note that C_φ is club by Lemmas 12.3 and 12.4.

  It remains only to show that C_φ reflects φ, that is, for every $\alpha \in C_\varphi$,

  $$\forall \overline{x} \in M_\alpha.(\exists y.\zeta(\overline{x}, y))^{M_\alpha} \Leftrightarrow (\exists y.\zeta(\overline{x}, y))^M.$$

  So, suppose $\alpha \in C_\varphi$ and $\overline{x} \in M_\alpha$.

  ($\Rightarrow$) We are given $(\exists y.\zeta(\overline{x}, y))^{M_\alpha}$, that is, there is some $y \in M_\alpha$ such that $\zeta^{M_\alpha}(\overline{x}, y)$ holds. Clearly $y, \overline{x} \in M$, and since $\alpha \in C_\zeta$, we conclude that $\zeta^M(\overline{x}, y)$ holds as well.

  ($\Leftarrow$) We have $(\exists y.\zeta(\overline{x}, y))^M$, that is, there is some $y \in M$ such that $\zeta^M(\overline{x}, y)$; we wish to show that there is some $y' \in M_\alpha$ such that $\zeta^{M_\alpha}(\overline{x}, y')$.

  Since $\alpha \in C_\varphi$, it is a limit ordinal, and therefore there is some $\beta < \alpha$ with $\overline{x} \in M_\beta$ (this follows from the definition of a cumulative hierarchy and the fact that $\overline{x}$ is finite). Furthermore, $G(\overline{x}) \leq F(\beta) < \alpha$. The existence of y implies that $G(\overline{x}) \neq 0$, so there is some $y' \in M_{G(\overline{x})} \subseteq M_\alpha$ such that $\zeta^M(\overline{x}, y')$ holds. Since $\alpha \in C_\zeta$, this implies that $\zeta^{M_\alpha}(\overline{x}, y')$ holds as well.

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Theorem 12.7. *There is no formula φ such that $Z+\varphi$ is consistent and extends ZF.*

Remark. Z here indicates ZF without Replacement; the above theorem shows that the infinite axiom schema of Replacement cannot be replaced by a finite one.

Proof. If $Z + \varphi$ extends ZF, then it derives the Reflection Principle, and in particular there is some least rank α that reflects φ and is a limit ordinal greater than ω (every club class contains arbitrarily large limit ordinals). That is,

$$Z + \varphi \vdash \exists \alpha. \lim(\alpha) \wedge \omega < \alpha \wedge \varphi^{V_\alpha} \wedge (\forall \beta < \alpha. \neg(\varphi^{V_\beta} \wedge \lim(\beta) \wedge \omega < \beta)).$$

But recall that V_α is a model of Z for every limit ordinal α greater than ω, so if γ is the least α whose existence is proven above, then

$$V_\gamma \models \exists \alpha. \lim(\alpha) \wedge \omega < \alpha \wedge \varphi^{V_\alpha}.$$

But this is a contradiction, since all the involved notions are absolute for V_γ, and so any element of V_γ satisfying the above would contradict the minimality of γ. However, to see the absoluteness of the above predicates will require some additional technical tools. To be continued. (Maybe.)

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