Lecture 16: The Constructible Hierarchy
March 23, 2009

13 The Constructible Hierarchy

Remark. We now return to Gödel's Constructible Hierarchy, L. Ultimately, we will show that

$$ZF \vdash ZF^L + AC^L + GCH^L$$

(where here "ZF" does not include the Axiom of Choice), thus proving the consistency of AC and GCH relative to that of ZF.

Definition 13.1. We define the constructible hierarchy L as follows:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= Def(L_\alpha) \\ L_\lambda &= \bigcup_{\beta<\lambda} L_\beta, \qquad \lim(\lambda). \end{aligned}$$

Intuitively, $Def(X)$ is the collection of sets definable in $\langle X, \in\rangle$ with parameters from X. But we will take some care to nail this down more rigorously.

Remark. We assume that our formal language has variables v_i, $i \in \omega$, and the usual connectives ($=$, $\in$, $\vee$, $\neg$, $\exists$). We now define a formal coding of formulas as sets (A "Gödel-setting" scheme, if you will.)

Definition 13.2. We define a "function" $Code$ sending formulas to sets. (Note it is only a function in a metaphorical sense, not a set-theoretic one, and is used only for convenience of notation.)

$$\begin{aligned} Code(v_i = v_j) &= \langle 0, i, j\rangle \\ Code(v_i \in v_j) &= \langle 1, i, j\rangle \\ Code(\varphi \vee \psi) &= \langle 2, Code(\varphi), Code(\psi)\rangle \\ Code(\neg\varphi) &= \langle 3, Code(\varphi)\rangle \\ Code(\exists v_i.\varphi) &= \langle 4, i, Code(\varphi)\rangle\,. \end{aligned}$$

Definition 13.3. We now define a relation Fm, which relates coded formulas u to their construction depth n and a sequence s of their subformulas.

$$
\begin{aligned}
Fm(u,n,s) \triangleq\ & n \in \omega \wedge Fn(s) \wedge \mathrm{dom}(s) = n+1 \wedge s(n) = u \\
& \wedge \forall k \leq n. \\
& \Big(\exists i,j < \omega . s(k) = Code(v_i = v_j) \\
& \vee \exists i,j < \omega . s(k) = Code(v_i \in v_j) \\
& \vee \exists l,m < k . s(k) = \langle 2, s(l), s(m) \rangle \\
& \vee \exists l < k . s(k) = \langle 3, s(l) \rangle \\
& \vee \exists l < k . \exists i < \omega . s(k) = \langle 4, i, s(l) \rangle \Big)
\end{aligned}
$$

Note that $Fn(x)$ is a predicate stating that x is a function. Then we also define $Fm(u) \triangleq \exists n. \exists s. Fm(u,n,s)$.

Remark. Finally, we define a satisfaction relation on formulas with respect to a set X. The idea is that if s_i is a coding for some subformula of u, then b_i will be the set of satisfiers of s_i, that is, the set of functions that assign free variables in s_i to elements of X in such a way that s_i is satisfied.

We want to be able to bound the domain of the satisfiers in b_i, but we can't just a priori pick some arbitrary limit. However, given a coding of a formula u, we know that the rank of u (denoted $\rho(u)$ in what follows) will be big enough, since it is certainly an upper bound on the indices of the free variables occuring in u (since each is embedded as an ordinal somewhere in u).

Definition 13.4. We define the relation Sat' on sets X, coded formulas u, and sequences of sets of satisfiers b as follows:

$$
\begin{aligned}
Sat'(X,u,b) \triangleq\ & \exists n. \exists s. Fm(u,n,s) \wedge Fn(b) \wedge \mathrm{dom}(b) = n+1 \\
& \wedge \mathrm{rng}(b) \subseteq {}^{\rho(u)}X \\
& \wedge \forall k < n+1. \\
& \Big((\exists i,j < \rho(u) . s(k) = Code(v_i = v_i) \wedge \forall t \in b(k) . t(i) = t(j)) \\
& \vee (\exists i,j < \rho(u) . s(k) = Code(v_i \in v_i) \wedge \forall t \in b(k) . t(i) \in t(j)) \\
& \vee (\exists l,m < k . s(k) = \langle 2, s(l), s(m) \rangle \wedge b(k) = b(l) \cup b(m)) \\
& \vee (\exists l < k . s(k) = \langle 3, s(l) \rangle \wedge b(k) = {}^{\rho(u)}X - b(l)) \\
& \vee (\exists l < k . \exists i < \rho(u) . s(k) = \langle 4, i, s(l) \rangle \wedge b(k) = \{\, t \mid \exists a \in X . t[i \mapsto a] \in b(l) \,\}) \Big)
\end{aligned}
$$

where $t[i \mapsto a] = t - \{\langle i, t(i) \rangle\} \cup \{\langle i, a \rangle\}$.

Definition 13.5. We can now define Sat as follows:

$$
Sat(X,u,t) \triangleq \exists b. \exists n \in \omega . Sat'(X,u,b) \wedge t \in b(n) \wedge \mathrm{dom}(b) = n+1.
$$

Definition 13.6. We define the notions of Σ_1, Π_1, and Δ_1-T formulas as follows:

- A formula φ is Σ_1 if there is some Δ_0 formula ψ such that $\varphi = \exists x.\psi$.
- A formula φ is Π_1 if there is some Δ_0 formula ψ such that $\varphi = \forall x.\psi$.
- φ is Δ_1-T for some theory T iff there is a Σ_1 formula ψ and a Π_1 formula χ such that
$$T \vdash \forall \overline{z}(\varphi(\overline{z}) \Leftrightarrow \chi(\overline{z}) \wedge \varphi(\overline{z}) \Leftrightarrow \psi(\overline{z})).$$

Lemma 13.7. *If φ is Δ_1-T then φ is absolute for transitive models of T.*

Proof. Suppose $M \subseteq M'$ are two transitive models of T, and we have some Δ_1-T formula $\varphi(\overline{z})$. We wish to show that $\varphi^M \Leftrightarrow \varphi^{M'}$ for all $\overline{z} \in M$.

($\Rightarrow$) Suppose $\varphi^M(\overline{z})$ holds. Then since M models T, we have $(\exists x.\psi(\overline{z}, x))^M$, that is, there exists $x \in M$ such that $\psi(\overline{z}, x)^M$. But we know that M is transitive and ψ is Δ_0, so $\psi(\overline{z}, x)^{M'}$ also holds (and $x \in M'$ since $M \subseteq M'$). Therefore, $\varphi^{M'}(\overline{z})$ holds.

($\Leftarrow$) Conversely, suppose $\varphi^{M'}(\overline{z})$ holds. Then we have $(\forall x.\chi(\overline{z}, x))^{M'})$. By a similar argument, since all $x \in M'$ are also in M, and χ is Δ_0, $(\forall x.\chi(\overline{z}, x))^M$ holds, and therefore so does $\varphi^M(\overline{z})$. ☒

Remark. Now we can give the sketch of an argument that Sat is Δ_1-ZF. We first note that Sat is Σ_1 as defined (it needs to be shown that Sat' is Δ_0). But we also note that by the way we constructed Sat', if some b exists which satisfies the definition of Sat, it is unique, and so $Sat(X, u, t)$ is equivalent to

$$\forall b.(\mathrm{dom}(b) = n + 1 \wedge Sat'(X, u, b) \Rightarrow t \in b(n)),$$

which is Π_1.