Definition 13.8. Following the previous lecture, we can now formally define \( \text{Def} \).

\[
\text{Def}(X) = \{ y \subseteq X \mid \exists \varphi. fv(\varphi) = \{ v_0, \ldots, v_n \} \\
\land \exists t. \text{dom}(t) = \{ v_0, \ldots, v_{n-1} \} \\
\land y = \{ a \in X \mid \text{Sat}(X, \varphi, t \cup \{(v_n, a)\}) \} \}
\]

Remark. Informally, we can think of this definition as

\[
D(X, y) \equiv \exists \varphi. \exists a. y = \{ a \in X \mid \text{Sat}(X, \varphi, t \cup \{(v_n, a)\}) \}.
\]

\[\text{Def}(x) = \{ y \mid D(X, y) \}.\]

Definition 13.9. A function \( F \) is \( \Sigma_1 \) iff the relation \( F(x) = y \) is \( \Sigma_1 \).

Lemma 13.10. If \( \text{dom}(F) \) is \( \Delta_1 \) and \( F \) is \( \Sigma_1 \), then \( F \) is \( \Delta_1 \).

Proof. Since \( F \) is \( \Sigma_1 \), we may suppose that \( F \) is given by some formula \( F(x, y) \equiv \exists z. \varphi(x, y, z) \).

Now consider the formula \( \chi(x, y) \equiv (\text{dom}(F)(x) \land \forall w.(\exists z. \varphi(x, w, z)) \Rightarrow w = y) \).

We claim that \( \chi \) is equivalent to \( F \), and that it is \( \Pi_1 \).

First, suppose \( F(x, y) \). Then there is some \( z \) for which \( \varphi(x, y, z) \), and \( (\text{dom}(F)(x) \) holds by definition. Now suppose there is some \( w \) for which \( \exists z. \varphi(x, w, z) \) holds. Then by definition, we have \( F(x, w) \). But since \( F \) is functional, \( w = y \).

Conversely, suppose \( \chi(x, y) \) holds. Then \( x \) is in the domain of \( F \), so there must be some \( y' \) for which \( F(x, y') \). But the second clause of \( \chi(x, y) \) implies that this \( y' \) must be equal to \( y \); hence \( F(x, y) \).

To see that \( \chi \) is \( \Pi_1 \), note that \( \text{dom}(F) \) is \( \Pi_1 \), and the \( \exists \) is on the left-hand side of an implication. More concretely, supposing that \( (\text{dom}(F)(x) \equiv \forall v. \psi(v, x) \right)

\[
\chi(x, y) \iff \forall v. \psi(v, x) \land \forall w. \neg(\exists z. \varphi(x, w, z)) \lor w = y \\iff \forall v. \psi(v, x) \land \forall w. \neg \varphi(x, w, z) \lor w = y \\iff \forall v. \forall w. \forall z. \psi(v, x) \land \neg \varphi(x, w, z) \lor w = y.
\]

Although this seems as though it has more than one unbounded quantifier, we could rewrite it as a single universal quantification over an ordered triple (this is known as “contraction”). Hence, \( \chi \) is \( \Pi_1 \).

Since \( F \) is \( \Sigma_1 \) and equivalent to a \( \Pi_1 \) formula, it is \( \Delta_1 \). \( \square \)

Remark. We remark that the class of \( \Sigma_1 \) formulas is closed under

- existential quantification,
• \( \land \) and \( \lor \) connectives, and
• bounded universal quantification.

The first two properties are obvious; the last is not.

A similar property holds for the class of \( \Pi_1 \) formulas.

Remark. The discussion of contraction at the end of the above proof shows that repetitions of the same unbounded quantifier are uninteresting. The above remark also shown that bounded quantifiers are not interesting. A real increase in complexity, however, comes from alternating unbounded quantifiers. \( \Sigma_2 \) is the class of formulas beginning with \( \exists \forall \); \( \Sigma_3 \) formulas begin with \( \exists \forall \exists \); and so on. \( \Pi_\alpha \) is similar.

Lemma 13.11. If \( G \) is \( \Sigma_1 \) and \( F \) is defined by transfinite recursion over \( G \), then \( F \) is \( \Delta_1 \).

Proof. Suppose we define \( F(\alpha) = G(F \upharpoonright \alpha) \) by transfinite recursion; formally, we define

\[
F(\alpha) = X \iff \exists f. \forall \beta \in \text{dom}(f). f(\beta) = G(f \upharpoonright \beta) \land f(\alpha) = X.
\]

Note that since \( G \) is \( \Sigma_1 \), so is \( f(\beta) = G(f \upharpoonright \beta) \land f(\alpha) = X \); hence so is \( F(\alpha) = X \) since the class of \( \Sigma_1 \) formulas is closed under bounded universal quantification and existential quantification. Also, the domain of \( F \) is the class of ordinals, which is \( \Delta_1 \) (in fact, it is \( \Delta_0 \)), so by Lemma 13.10 \( F \) is \( \Delta_1 \).

Theorem 13.12. \( L \) is \( \Delta_1 \).

Proof. \( L \) is defined by transfinite recursion over a \( \Sigma_1 \) function (it is left as an exercise to check that \( \text{Def} \) is \( \Sigma_1 \)).

Corollary 13.13. \( L \) is absolute for transitive models of \( \text{ZF} \).

Definition 13.14. The order of a set \( X \), denoted \( \text{od}(X) \), is the least \( \alpha \) such that \( X \in L_{\alpha+1} \). (It is not yet clear that this is well-defined for all sets, although it turns out that it is.)

Definition 13.15. A class \( M \) is almost universal iff for every \( X \subseteq M \), then there is some \( Y \in M \) for which \( X \subseteq Y \).

Lemma 13.16. If \( M \) contains \( \text{On} \) (the class of ordinals) and is transitive and almost universal, and \( (\text{Sep})^M \) (that is, \( M \) satisfies the axiom of Separation), then \( (\text{ZF})^M \).

Proof. Deferred to the next lecture.

Lemma 13.17. \( L \) satisfies the conditions of Lemma 13.16.

Proof. We show each of the conditions in turn.
• \(L\) is transitive, that is, \(L_\alpha\) is transitive for all \(\alpha\). Since a union of transitive sets is transitive, it suffices to show that \(Def(X)\) is transitive if \(X\) is.

Suppose \(X\) is transitive, and that \(y \in Def(X)\). Thus \(y \subseteq X\). We want to show that \(y \subseteq Def(X)\). Suppose \(z \in y\), and consider the formula \(\varphi(w) = w \in z\). Then the set \(\{w \mid (X, \varepsilon) \models \varphi(w)\} \in Def(x)\); but since \(X\) is transitive, every member of \(z\) is a member of \(X\), so this set is equal to \(z\), and \(y \subseteq Def(X)\).

• To show that \(L\) contains \(On\), we will in fact show the stronger statement that \(L_\alpha \cap On = \alpha\), for all \(\alpha\). The proof is by induction on \(\alpha\). The base case is easily verified.

In the limit case, \(On \cap L_\lambda = On \cap \bigcup_{\beta < \lambda} L_\beta = \bigcup_{\beta < \lambda} (On \cap L_\beta) = \bigcup_{\beta < \lambda} \beta = \lambda\).

In the successor case, suppose \(On \cap L_\alpha = \alpha\). Since \(L\) is cumulative, we need only show that \(\alpha \in L_{\alpha + 1}\); to see this, consider the defining formula \(On(\beta)\) over \(L_\alpha\). Since \(On\) is \(\Delta_0\), it is absolute, so it picks out exactly the elements of \(\alpha\).

• \(L\) is almost universal. Given \(Y \subseteq L\), consider

\[
\beta = \sup \{od(x) + 1 \mid x \in Y\}.
\]

Then \(Y \subseteq L_\beta \subseteq L_{\beta + 1}\).

• \(L\) satisfies Separation. Suppose \(x \in L\), and consider the set

\[
s = \{y \in x \mid \varphi^L(y)\}.
\]

We must show that \(s\) is in \(L\) also. Consider \(\beta = od(x)\). By the Reflection Principle, there is some \(\alpha > \beta\) such that

\[
\forall y \in L_\alpha, \varphi^L(y) \iff \varphi^{L_\alpha}(y).
\]

But since every \(y \in x\) is also in \(L_\alpha\), this means that \(s \in L_{\alpha + 1}\); we may take the defining formula to be \(\varphi(y) \land y \in x\).