Lecture 17: The Constructible Hierarchy, Part II

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Definition 13.8. Following the previous lecture, we can now formally define Def.

$$\begin{aligned} Def(X) = \{\, y \subseteq X \mid & \exists\varphi.fv(\varphi) = \{v_0, \dots v_n\} \\ & \wedge \exists t.\operatorname{dom}(t) = \{v_0, \dots v_{n-1}\} \\ & \wedge y = \{\, a \in X \mid Sat(X, \varphi, t \cup \{\langle v_n, a\rangle\})\,\}\,\} \end{aligned}$$

Remark. Informally, we can think of this definition as

$$\begin{aligned} D(X, y) &\triangleq \exists\varphi.\exists\overline{a}.y = \{\, a \in X \mid \langle X, \in\rangle \models \varphi[\overline{a}, a]\,\} \\ Def(x) &= \{\, y \mid D(X, y)\,\}. \end{aligned}$$

Definition 13.9. A function F is Σ_1 iff the relation $F(x) = y$ is Σ_1.

Lemma 13.10. *If* $\operatorname{dom}(F)$ *is* Δ_1 *and* F *is* Σ_1*, then* F *is* Δ_1*.*

Proof. Since F is Σ_1, we may suppose that F is given by some formula $F(x, y) \triangleq \exists z.\varphi(x, y, z)$.

Now consider the formula

$$\chi(x, y) \triangleq (\operatorname{dom} F)(x) \wedge \forall w.(\exists z.\varphi(x, w, z)) \Rightarrow w = y.$$

We claim that χ is equivalent to F, and that it is Π_1.

First, suppose $F(x, y)$. Then there is some z for which $\varphi(x, y, z)$, and $(\operatorname{dom} F)(x)$ holds by definition. Now suppose there is some w for which $\exists z.\varphi(x, w, z)$ holds. Then by definition, we have $F(x, w)$. But since F is functional, $w = y$.

Conversely, suppose $\chi(x, y)$ holds. Then x is in the domain of F, so there must be some y' for which $F(x, y')$. But the second clause of $\chi(x, y)$ implies that this y' must be equal to y; hence $F(x, y)$.

To see that χ is Π_1, note that $\operatorname{dom} F$ is Π_1, and the $\exists$ is on the left-hand side of an implication. More concretely, supposing that $(\operatorname{dom} F)(x) \triangleq \forall v.\psi(v, x)$,

$$\begin{aligned} \chi(x, y) &\iff \forall v.\psi(v, x) \wedge \forall w.\neg(\exists z.\varphi(x, w, z)) \vee w = y \\ &\iff \forall v.\psi(v, x) \wedge \forall w.\forall z.\neg\varphi(x, w, z) \vee w = y \\ &\iff \forall v.\forall w.\forall z.\psi(v, x) \wedge \neg\varphi(x, w, z) \vee w = y. \end{aligned}$$

Although this seems as though it has more than one unbounded quantifier, we could rewrite it as a single universal quantification over an ordered triple (this is known as "contraction"). Hence, χ is Π_1.

Since F is Σ_1 and equivalent to a Π_1 formula, it is Δ_1. ∎

Remark. We remark that the class of Σ_1 formulas is closed under

- existential quantification,

- $\wedge$ and $\vee$ connectives, and
- bounded universal quantification.

The first two properties are obvious; the last is not.

A similar property holds for the class of Π_1 formulas.

Remark. The discussion of contraction at the end of the above proof shows that repetitions of the same unbounded quantifier are uninteresting. The above remark also shown that bounded quantifiers are not interesting. A real increase in complexity, however, comes from alternating unbounded quantifiers. Σ_2 is the class of formulas beginning with $\exists\forall$; Σ_3 formulas begin with $\exists\forall\exists$; and so on. Π_n is similar.

Lemma 13.11. *If G is Σ_1 and F is defined by transfinite recursion over G, then F is Δ_1.*

Proof. Suppose we define $F(\alpha) = G(F \restriction \alpha)$ by transfinite recursion; formally, we define

$$F(\alpha) = X \iff \exists f . \forall \beta \in \mathrm{dom}(f) . f(\beta) = G(f \restriction \beta) \wedge f(\alpha) = X.$$

Note that since G is Σ_1, so is $f(\beta) = G(f \restriction \beta) \wedge f(\alpha) = X$; hence so is $F(\alpha) = X$ since the class of Σ_1 formulas is closed under bounded universal quantification and existential quantification. Also, the domain of F is the class of ordinals, which is Δ_1 (in fact, it is Δ_0), so by Lemma 13.10 F is Δ_1. ∎

Theorem 13.12. *L is Δ_1.*

Proof. L is defined by transfinite recursion over a Σ_1 function (it is left as an exercise to check that Def is Σ_1). ∎

Corollary 13.13. *L is absolute for transitive models of ZF.*

Definition 13.14. The *order* of a set X, denoted $od(X)$, is the least α such that $X \in L_{\alpha+1}$. (It is not yet clear that this is well-defined for all sets, although it turns out that it is.)

Definition 13.15. A class M is *almost universal* iff for every $X \subseteq M$, then there is some $Y \in M$ for which $X \subseteq Y$.

Lemma 13.16. *If M contains On (the class of ordinals) and is transitive and almost universal, and $(Sep)^M$ (that is, M satisfies the axiom of Separation), then $(ZF)^M$.*

Proof. Deferred to the next lecture. ∎

Lemma 13.17. *L satisfies the conditions of Lemma 13.16.*

Proof. We show each of the conditions in turn.

- L is transitive, that is, L_α is transitive for all α. Since a union of transitive sets is transitive, it suffices to show that $Def(X)$ is transitive if X is.

  Suppose X is transitive, and that $y \in Def(X)$. Thus $y \subseteq X$. We want to show that $y \subseteq Def(X)$. Suppose $z \in y$, and consider the formula $\varphi(w) = w \in z$. Then the set $\{ w \mid \langle X, \in \rangle \models \varphi(w) \} \in Def(x)$; but since X is transitive, every member of z is a member of X, so this set is equal to z, and $y \subseteq Def(X)$.

- To show that L contains On, we will in fact show the stronger statement that $L_\alpha \cap On = \alpha$, for all α. The proof is by induction on α. The base case is easily verified.

  In the limit case, $On \cap L_\lambda = On \cap \bigcup_{\beta<\lambda} L_\beta = \bigcup_{\beta<\lambda}(On \cap L_\beta) = \bigcup_{\beta<\lambda} \beta = \lambda$.

  In the successor case, suppose $On \cap L_\alpha = \alpha$. Since L is cumulative, we need only show that $\alpha \in L_{\alpha+1}$; to see this, consider the defining formula $On(\beta)$ over L_α. Since On is Δ_0, it is absolute, so it picks out exactly the elements of α.

- L is almost universal. Given $Y \subseteq L$, consider

  $$\beta = \sup\{od(x) + 1 \mid x \in Y\}.$$

  Then $Y \subseteq L_\beta \in L_{\beta+1}$.

- L satisfies Separation. Suppose $x \in L$, and consider the set

  $$s = \{ y \in x \mid \varphi^L(y) \}.$$

  We must show that s is in L also. Consider $\beta = od(x)$. By the Reflection Principle, there is some $\alpha > \beta$ such that

  $$\forall y \in L_\alpha . \varphi^L(y) \iff \varphi^{L_\alpha}(y).$$

  But since every $y \in x$ is also in L_α, this means that $s \in L_{\alpha+1}$; we may take the defining formula to be $\varphi(y) \wedge y \in x$.

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