Lecture 18: The Constructible Hierarchy, Part III
March 30, 2009

Proof of Lemma 13.16. We are given a transitive, almost universal class M which contains On and satisfies Sep; we wish to show that M satisfies ZF.

- Ext^M since M is transitive.
- Reg^M since M is a class.
- $Pair^M$. Suppose $x \in M$ and $y \in M$. By pairing (in the universe) there is some $z = \{x, y\} \subseteq M$. Since M is almost universal, there is some $u \in M$ such that $z \subseteq u$. Now consider the set $\{\, w \in u \mid w = x \vee w = y \,\}$. This set is in M since M satisfies Separation; but this set is precisely the pair $\{x, y\}$ in M, since $w = x \vee w = y$ is Δ_0.
- $Union^M$. Let $M(x)$. Then by the union axiom, $\exists \bigcup x . \forall z . z \in \bigcup x \Leftrightarrow \exists b \in x . z \in b$.

  Note that $y \in \bigcup x \implies y \in M$, since M is transitive; so by the almost universality of M, we conclude there is some $u \in M$ for which $\bigcup x \subseteq u$. Now consider the formula $\varphi(y) \triangleq \exists b \in x . y \in b$.

  Note that Sep^M expands to

  $$\forall x \in M . \exists y \in M . \forall z \in M . z \in y \Leftrightarrow z \in x \wedge \varphi^M(z).$$

  So we may conclude that there is some $p \in M$ such that $\forall z \in M . z \in p \Leftrightarrow z \in u \wedge \varphi^M(z)$, that is,

  $$p = \{\, z \in u \mid \varphi^M(z) \,\}.$$

  We want to show the union axiom relativized to M, that is, $\exists q \in M . \forall z \in M . z \in q \Leftrightarrow \varphi^M(z)$. We claim that p witnesses this formula. The ($\Rightarrow$) direction holds by definition of p. The ($\Leftarrow$) direction holds since φ is Δ_0, so $\varphi^M(z)$ implies $\varphi(z)$ (since $z \in M$) and $\varphi(z)$ states that $z \in \bigcup x$; and $\bigcup x \subseteq u$.
- $Powerset^M$. Let $M(x)$. Then by the power set axiom, $\mathcal{P}(x)$ exists in the universe. Note that this may *not* be the power set of x in M, since M does not necessarily contain all subsets of x. We want to show that

  $$v = \{\, w \in \mathcal{P}(x) \mid M(w) \,\}$$

  is in M. Since M is almost universal, there is some $u \in M$ for which $v \subseteq u$; then by comprehension in M we may form the set $\{\, z \in u \mid z \subseteq x \,\}$; this set is precisely v ($\subseteq$ is Δ_0).
- $Infinity^M$. We stipulated that $On \subseteq M$, so in particular we have $\omega \in M$, and ω is absolute.

- *Replacement*^{M}. Suppose $\varphi(x, y)$ is a functional relation in M, that is, $\forall x \in M.\exists! y \in M. \wedge \varphi^M(x, y)$. We wish to show

$$\forall w \in M.\exists u \in M.\forall x \in w.\exists y \in u.\varphi^M(x, y).$$

  This is the relativization to M of a weak form of the axiom of replacement. It shows that u contains the image of w under φ; we can use separation to construct the exact image of w under φ.

  Let $w \in M$; the φ-image of w exists in the universe, call it v. Then by almost universality of M, there is some $u' \in M$ for which $v \subseteq u'$; then we are done. ∎

Corollary 13.18. *ZF*^{L}.

Definition 13.19. A model M of ZF is an *inner model* iff M is a transitive class containing On.

Remark. We have seen previously that there is a Δ_1-ZF relation C such that $C(\alpha, x)$ iff $x = L_\alpha$. Hence $C(\alpha, x)$ is absolute for inner models of ZF.

Lemma 13.20. *If M is an inner model of ZF, then $L^M = L$. (Where $L^M = \{ y \mid (\exists \alpha, x.\ C(\alpha, x) \wedge y \in x)^M \}$.)*

Proof. ??? ∎

Corollary 13.21. $ZF \vdash (V = L)^L$.

Proof. $(V = L)^L = (V^L = L^L) = (L = L)$. ∎

Corollary 13.22. *L is the smallest inner model of ZF.*

Proof. Any inner model M contains $L^M = L$. ∎

Remark. Recall that we are in the middle of trying to prove

$$ZF \vdash ZF^L + AC^L + GCH^L,$$

by showing that

$$ZF + \text{“}V = L\text{”} \vdash AC + GCH$$

and

$$ZF \vdash ZF^L + (V = L)^L.$$

We have now shown the second part; it remains only to show that AC and GCH hold in ZF + “$V = L$”.

Theorem 13.23. $ZF + (V = L) \vdash AC$.

Proof. There is a definable relation $<_L$ which is a global well-ordering of L (this is bizarre). Define $<_{L,\alpha}$ inductively as follows.

- $<_{L,0}$ is the empty relation.

- At limit stages, we of course take the union of all previous stages.
- Now we define $<_{L,\alpha+1}$ in terms of $<_{L,\alpha}$. Note that every $x \in L_{\alpha+1}$ is a subset of L_α defined in terms of some $n \in \omega$, some $\overline{y} \in L_\alpha^n$, and some first-order formula φ. We can order formulas using a Gödel numbering. We can also order tuples lexicographically, so given an ordering of L_α, we can order elements of L_α^n. We now order $L_{\alpha+1}$ in the obvious way: for each $x \in L_{\alpha+1}$, choose (in some canonical order) the least n, least formula φ, and least tuple that define it. Also, we stipulate that everything at stage α comes before everything first arising at stage $\alpha + 1$.

We then take $x <_L y$ to mean that there exists some α for which $x <_{L,\alpha} y$.

Hence every set in L has a well-ordering, so AC holds. (But moreover, the entire universe is well-ordered! This gives an intuitive reason to believe that $V = L$ is not really true in a Platonic sense.) ⊠