Lecture 23: Independence of CH, part IV

April 15, 2009

Remark. In the previous lecture, we defined the generic extension $M[G]$ of any ctm M with respect to a set G which is $\mathbb{P}$-generic over M. Today, we will begin to verify that it has the required properties. In particular:

- $M \subseteq M[G]$. (We showed this in the previous lecture.)
- $G \in M[G]$.
- $M[G]$ is transitive.
- $o(M) = o(M[G])$.
- $M[G]$ is a ctm.
- $M[G]$ is the least extension of M with these properties.

Lemma 14.21. *$G \in M[G]$.*

Proof. Consider the set
$$\Delta = \{\, \langle \dot{p}, p \rangle \mid p \in \mathbb{P} \,\}.$$
We have already seen that $\dot{x} \in M$ whenever $x \in M$, so $\Delta \in M$ by pairing and replacement (M is a ctm). Also, Δ is clearly a $\mathbb{P}$-name. But $\mathrm{val}(\Delta, G) = G$, so we conclude that $G \in M[G]$. ∎

Lemma 14.22. *$M[G]$ is the least extension of M with the required properties.*

Proof. Suppose there is some ctm N such that $M \subseteq N$ and $G \in N$. $M^{\mathbb{P}} \subseteq N$ since $M \subseteq N$, so $\mathrm{val}(\tau, G) \in N$ for all $\tau \in M^{\mathbb{P}}$ (val is definable in $M[G]$, and absolute since it is defined by recursion). Therefore, $M[G] \subseteq N$. ∎

Remark. From now on we will use the abbreviation τ_G in place of $\mathrm{val}(\tau, G)$.

Lemma 14.23. *$M[G]$ is transitive.*

Proof. Suppose $x \in M[G]$ and $y \in x$. By definition of $M[G]$, there is some $\tau \in M^{\mathbb{P}}$ for which $x = \tau_G$. Expanding out the definition of τ_G, we have
$$x = \tau_G = \{\, \sigma_G \mid \exists p \in G.\, \langle \sigma, p \rangle \in \tau \,\}.$$
Therefore $y = \sigma_G$ for some $\sigma \in V^{\mathbb{P}}$, but since M is transitive, $\sigma \in M$ as well (since τ is). Hence $y \in M[G]$. ∎

Lemma 14.24. *$o(M) = o(M[G])$.*

Proof. $o(M) \le o(M[G])$ follows directly from the fact that $M \subseteq M[G]$.

To show that $o(M[G]) \le o(M)$, we show that $\mathrm{rank}(\tau_G) \le \mathrm{rank}(\tau)$, by structural induction on τ. If $\tau = \emptyset$, then $\mathrm{rank}(\tau_G) = \mathrm{rank}(\emptyset) = 0$.

In the inductive case,

$$\begin{aligned}
\mathrm{rank}(\tau_G) &= \mathrm{rank}(\{\, \sigma_G \mid \exists p \in G.\, \langle \sigma, p \rangle \in \tau \,\}) \\
&\le \mathrm{rank}(\{\, \sigma_G \mid \langle \sigma, p \rangle \in \tau \,\}) \\
&\le \mathrm{rank}(\{\, \sigma \mid \langle \sigma, p \rangle \in \tau \,\}) && \text{(IH)} \\
&\le \mathrm{rank}(\{\, \langle \sigma, p \rangle \mid \langle \sigma, p \rangle \in \tau \,\}) \\
&= \mathrm{rank}(\tau).
\end{aligned}$$

Now we note that if $\alpha \in M[G]$, then there is some $\tau \in M$ for which $\tau_G = \alpha$, and $\alpha = \mathrm{rank}(\alpha) \le \mathrm{rank}(\tau) = \beta$, and that $\mathrm{rank}(\tau) \in M$ whenever $\tau \in M$. (??) ∎

Remark. It remains to show that $M[G]$ is a ctm; but before we do that, we talk about the method of forcing, and use it to prove the Approximation Lemma (Lemma 14.15).

Definition 14.25. Let M be a ctm and $\mathbb{P} \in M$ a poset with a maximal element. Suppose $\varphi(x_1, \dots, x_n)$ is some formula and $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$. Then p *forces* $\varphi(\tau_1, \dots, \tau_n)$, written

$$p \Vdash_{M,\mathbb{P}} \varphi(\tau_1, \dots, \tau_n),$$

iff for every G which is a $\mathbb{P}$-generic extension over M with $p \in G$,

$$M[G] \models \varphi(\tau_{1G}, \dots, \tau_{nG}).$$

Remark. Often M and $\mathbb{P}$ will be clear from the context and we omit the subscripts on $\Vdash$.

Remark. We now state two essential (and somewhat surprising) results about forcing; their proofs will be put off until later.

Theorem 14.26 (Truth)**.** *$M[G] \models \varphi(\tau_{1G}, \dots, \tau_{nG})$ if and only if there is some $p \in G$ for which $p \Vdash \varphi(\tau_1, \dots, \tau_n)$.*

Theorem 14.27 (Definability)**.** *For every $\varphi(x_1, \dots, x_n)$, there is a formula denoted*

$$p \Vdash^* \varphi(x_1, \dots, x_n)$$

such that for all $\tau_1, \dots, \tau_n$, $p \Vdash \varphi(\tau_1, \dots, \tau_n)$ if and only if $M \models (p \Vdash^* \varphi(\tau_1, \dots, \tau_n))$.

Remark. In other words, the notion of forcing is definable within M itself. This is rather surprising, since the definition of forcing quantifies over all generic extensions, which are not elements of M!

Lemma 14.28 (Preservation of forcing)**.** *For all formulas φ and $s, t \in \mathbb{P}$, if $s \le t$ and $t \Vdash \varphi$, then $s \Vdash \varphi$.*

Proof. Suppose $s \nVdash \varphi$—that is, there is some G $\mathbb{P}$-generic over M with $s \in G$ and $M[G] \nvDash \varphi$. But since G is a filter, it is upward closed; hence $s \in G$ implies $t \in G$, which is a contradiction since $t \Vdash \varphi$. ⊠

Remark. We now return to prove the Approximation Lemma (Lemma 14.15).

Proof of Lemma 14.15. Let $\tau \in M^{\mathbb{P}}$ such that $\tau_G = f$. By Theorem 14.26 (Truth), there is some $p \in G$ such that $p \Vdash \tau : \dot{x} \to \dot{y}$.

Now, for each $a \in X$, define

$$F(a) = \{\, b \mid \exists q \leq p.\ q \Vdash \tau(\dot{a}) = \dot{b} \,\}.$$

Then by definability of forcing in M (Theorem 14.27) and the fact that M is a ctm, we have that $F \in M$.

Now suppose $f(a) = b$; we wish to show that $b \in F(a)$. Since $f(a) = b$, in particular we have that $M[G] \models \tau_G(a) = b$. Hence, by Truth, there is some $r \in G$ such that $r \Vdash \tau(\dot{a}) = \dot{b}$. Since G is a filter and $p, r \in G$, there is some $q \in G$ for which $q \leq p$ and $q \leq r$. By Lemma 14.28, $q \Vdash \tau(\dot{a}) = \dot{b}$. But then $b \in F(a)$ by definition.

Finally, we show that $F(a)$ is countable in M for every $a \in X$. Since M is a ctm, it satisfies AC, so there is a choice function $g : F(a) \to G$ such that $g(b) \leq p$ and $g(b) \Vdash \tau(\dot{a}) = \dot{b}$ for each $b \in F(a)$; that is, for each b, g picks a witness of the fact that $b \in F(a)$. (We note that for each b, the set of q which witness $b \in F(a)$ is in M by definability of forcing and the fact that M is a ctm.)

We claim that for any two distinct $b, b' \in F(a)$, $g(b) \perp g(b')$. (Note that this also implies that g is injective.) To see this, suppose $b \neq b'$ and $g(b) \top g(b')$. Then since G is a filter, there exists some r for which $r \leq g(b)$ and $r \leq g(b')$. But then by preservation of forcing,

$$\begin{aligned}
&r \Vdash \tau : \dot{x} \to \dot{y} \text{ (since } r \leq g(b) \leq p),\\
&r \Vdash \tau(\dot{a}) = \dot{b}, \text{ and}\\
&r \Vdash \tau(\dot{a}) = \dot{b'},
\end{aligned}$$

which is a contradiction since we assumed that $b \neq b'$.

Therefore, $g[F(a)]$ is an antichain, and hence countable in M since $\mathbb{P}$ has the ccc in M by assumption. Therefore, since g is injective, $F(a)$ is countable in M. ⊠

Remark. We now know, by Lemma 14.16, that any extension of a ctm M defined with respect to a $FP(\aleph_2 \times \omega, 2)$-generic set doesn't collapse cardinals.

We also note the general shape of the preceding proof: we went from some combinatorial property of a partial order $\mathbb{P}$ (here, the ccc property of $FP(X, Y)$) to a property of $\mathbb{P}$-generic extensions of a ctm M. This is typical of forcing arguments, although in general the combinatorial proerties may be much more complicated, and the proofs correspondingly more difficult.