Lecture 24: Independence of CH, part V

April 20, 2009

Lemma 14.29. *$M[G]$ is a ctm of ZFC.*

Proof. We show that $M[G]$ satisfies each axiom of ZFC.

- Extensionality. Follows easily from transitivity of $M[G]$.
- Regularity. Trivial.
- Pairing. Let $x, y \in M[G]$; then there exist $\tau, \sigma \in M^{\mathbb{P}}$ with $\tau_G = x$ and $\sigma_G = y$. Now consider the set

  $$\delta = \{\langle \tau, 1_{\mathbb{P}} \rangle, \langle \sigma, 1_{\mathbb{P}} \rangle\}.$$

  It is easy to see that $\delta_G = \{\tau_G, \sigma_G\} = \{x, y\}$. But note that $\delta \in M^{\mathbb{P}}$: it is a $\mathbb{P}$-name by construction, and is in M since M is a ctm.

- Union. Suppose $a \in M[G]$. We wish to show that there is some $b \in M[G]$ which contains $\bigcup a$ as a subset (we can then appeal to Separation in $M[G]$, which we will show later).

  Since $a \in M[G]$, there is some $\tau \in M^{\mathbb{P}}$ with $\tau_G = a$. Let $\pi = \bigcup \mathrm{dom}(\tau)$; this is a set which contains the $\mathbb{P}$-names of all elements of τ_G (and possibly some extra ones whose corresponding conditions are not in G). $\pi \in M$ since M is a ctm; $\pi \in V^{\mathbb{P}}$ by construction ($\mathrm{dom}(\tau)$ is a set of $\mathbb{P}$-names, so $\bigcup \mathrm{dom}(\tau)$ is a subset of $V^{\mathbb{P}} \times \mathbb{P}$). Hence $\pi \in M^{\mathbb{P}}$, so $\pi_G \in M[G]$.

  We claim that $\bigcup a \subseteq \pi_G$. To see this, let $c \in a$; then $c = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$. Therefore $\sigma \subseteq \pi$, so $\sigma_G \subseteq \pi_G$.

- Separation. Let $\sigma \in M^{\mathbb{P}}$ and let φ be a formula (it may have multiple parameters, but we omit them in the following proof), and define

  $$c = \{\, a \in \sigma_G \mid M[G] \models \varphi[a] \,\}.$$

  We wish to show that $c \in M[G]$, which we will do by finding a suitable $\mathbb{P}$-name for c.

  We claim that a suitable $\mathbb{P}$-name is

  $$\rho = \{\, \langle \pi, p \rangle \in \mathrm{dom}(\sigma) \times \mathbb{P} \mid p \Vdash \pi \in \sigma \wedge \varphi(\pi) \,\}.$$

  We first note that $\rho \in M$ by separation in M and definability of $\Vdash$ (Theorem 14.27); ρ is clearly a $\mathbb{P}$-name by construction. Now we must show that $\rho_G = c$.

  - ($\rho_G \subseteq c$). Suppose $x \in \rho_G$, so there is some $\langle \pi, p \rangle \in \rho$ such that $x = \pi_G$ and $p \Vdash \pi \in \sigma \wedge \varphi(\pi)$ and $p \in G$. Then by definition of forcing, $\pi_G \in \sigma_G$ and $M[G] \models \varphi[\pi_G]$. Hence $x = \pi_G \in c$ by definition of c.

– ($c \subseteq \rho_G$). Suppose $a \in c$, that is, $a \in \sigma_G$ and $M[G] \models \varphi[a]$. Then there is some $\pi \in M^{\mathbb{P}}$ such that $\pi_G = a$. So by Truth (Theorem 14.26) we may pick $p \in G$ such that $p \Vdash \pi \in \sigma \wedge \varphi(\pi)$. Then $\langle \pi, p \rangle \in \rho$, so $a = \pi_G \in \rho_G$.

- Replacement. At this point, we introduce the axiom schema of Collection:

$$\forall x.\exists y.\forall z \in x.(\exists w.\varphi(z, w) \Rightarrow \exists w \in y.\varphi(z, w)).$$

Intuitively, this states that we can collect elements in the image of any set x under any partial relation φ into a set y (which may also contain other stuff). This implies the axiom schema of Replacement: we may take φ to be a functional relation, and then given a set y witnessing Collection, we may use Separation to yield a set which is exactly the image $\varphi[x]$.

It turns out that Collection is also a theorem of ZF, via the reflection principle.

Now suppose we have some $x = \sigma_G$; we wish to exhibit a ρ for which

$$M[G] \models \forall z \in \sigma_G.(\exists w.\varphi(z, w) \Rightarrow \exists w \in \rho_G.\varphi(z, w)). \tag{2}$$

Let $S \in M$ such that

$$\begin{aligned} M \models \forall \pi \in \mathrm{dom}(\sigma).\forall p \in \mathbb{P}.(\exists \mu.M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu) \\ \Rightarrow (\exists \mu \in S).p \Vdash \varphi(\pi, \mu)). \end{aligned}$$

It is not *a priori* clear that such an S exists. If $M^{\mathbb{P}}$ were a set, we could just take $S = M^{\mathbb{P}}$, but $M^{\mathbb{P}}$ may be a proper class. However, such an S does exist, which we can show as follows (note that in the following, all our reasoning is taking place *inside* M). By Reflection in M, there is a closed unbounded class of ordinals α which simultaneously reflect the two formulae

$$\exists \mu.M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu)$$

and

$$M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu),$$

that is,

$$\begin{aligned} \forall \pi \in \mathrm{dom}(\sigma).\forall p \in \mathbb{P}.\big(\exists \mu.M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu) \\ \Leftrightarrow [\exists \mu.M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu)]^{V_\alpha}\big), \end{aligned} \tag{3}$$

and

$$\begin{aligned} \forall \pi \in \mathrm{dom}(\sigma).\forall p \in \mathbb{P}.\forall \mu.\big(M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu) \\ \Leftrightarrow [M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu)]^{V_\alpha}\big). \end{aligned} \tag{4}$$

So, we may pick such an α large enough so that $\mathrm{dom}(\sigma) \in V_\alpha$ and $\mathbb{P} \in V_\alpha$.

We then let $S = M^{\mathbb{P}} \cap V_\alpha$, and claim that S has the required property. Given some $\pi \in \text{dom}(\sigma)$ and $p \in \mathbb{P}$, suppose there exists some $\mu \in M^{\mathbb{P}}$ for which $p \Vdash \varphi(\pi, \mu)$. Then by equation (3) there is some $\mu \in V^\alpha$ which satisfies $[M^{\mathbb{P}}(\mu) \wedge p \Vdash \varphi(\pi, \mu)]^{V_\alpha}$; but then by equation (4) μ also satisfies this condition in the universe, so $\mu \in S$ and $p \Vdash \varphi(\pi, \mu)$, exactly the required property of S.

Now let $\rho = S \times \{1_{\mathbb{P}}\}$, so $\rho_G = \{ \mu_G \mid \mu \in S \}$ (since G is a filter). Now we must show that ρ satisfies equation (2).

To this end, let $z \in \sigma_G$ and $\varphi^{M[G]}(z, w)$ for some $w \in M[G]$. We must find some $w' \in \rho_G$ for which $\varphi^{M[G]}(z, w')$.

Since $z \in \sigma_G$, $z = \pi_G$ for some $\pi \in \text{dom}(\sigma)$. We know that $M[G] \models \exists w.\varphi(\pi_G, w)$, so there must be some μ for which $M[G] \models \varphi(\pi_G, \mu_G)$. Then by Truth there is some $p \in G$ such that $p \Vdash \varphi(\pi, \mu)$. Then by the property of S, there is some $\mu' \in S$ such that $p \Vdash \varphi(\pi, \mu')$, and $\mu'_G \in \rho_G$. ⊠

Remark. We are not quite done; in the next lecture we will cover Powerset and Choice. But now, a small digression about the axiom schema of Collection.

Definition 14.30. *Kripke-Platek set theory* is the axiomatic system with Extensionality, Regularity, Pairing, Union, and all Δ_0 instances of Separation and Collection.

Remark. It is easy to see that $V_\omega \models KP$, since it models ZF $-$ Infinity. KP $+$ Infinity is a nice system, too.

Definition 14.31. An ordinal α is *admissible* iff $L_\alpha \models KP$.

Remark. Admissible ordinals "are those which support a nice notion of computability."

Definition 14.32. $R \subseteq \omega \times \omega$ is *recursive* iff c_R, the characteristic function of R, is Turing-computable. An ordinal α is recursive iff it is the order type of some recursive $R \subseteq \omega \times \omega$.

Definition 14.33. ω_1^{CK}, the *Church-Kleene ordinal*, is the least non-recursive ordinal.

$(\omega_1^{CK})^f$ is the least non-(recursive)^f ordinal, where $f \in \omega \to 2$ and (recursive)^f means Turing-computable given an f-oracle.

Theorem 14.34. *If α is a countable ordinal greater than ω, then α is admissible iff $\alpha = (\omega_1^{CK})^f$ for some $f \in \omega \to 2$.*

Remark. The proof is omitted.

We note that ω_1^{CK} is, in fact, the set of all recursive ordinals, so in particular it must be countable (since there are countably many Turing machines).