Lecture 26: Ramsey cardinals

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Definition 15.4. κ is *weakly compact* iff κ is uncountable and $\kappa \to (\kappa)^2_2$.

Lemma 15.5. *If κ is weakly compact, then $\kappa \to (\kappa)^2_\mu$ for every $\mu < \kappa$.*

Proof. The proof is a problem on the final exam. ∎

Lemma 15.6. *If κ is weakly compact, then κ is strongly inaccessible.*

Proof. We must show that κ is regular, and that it is a strong limit.

- κ is regular. Suppose otherwise; then let $\lambda < \kappa$ and $\{ \gamma_\alpha \mid \alpha < \lambda \} \subseteq \kappa$ an increasing sequence with $\sup\{ \gamma_\alpha \mid \alpha < \lambda \} = \kappa$. Without loss of generality, assume $\gamma_0 = 0$.

  We note that γ naturally induces a partition of κ into λ many segments. Now, define

  $$f(\{\delta, \zeta\}) = \begin{cases} 0 & \exists \alpha \geq 0 . \delta, \zeta \in [\gamma_\alpha, \gamma_{\alpha+1}), \\ 1 & \text{otherwise.} \end{cases}$$

  f induces a 2-partition on $[\kappa]^2$; hence, since κ is weakly compact, there must be some set $Y \subseteq \kappa$ of cardinality κ where f is constant on $[Y]^2$.

  Since f is constant on $[Y]^2$, there are two cases to consider. First, we could have $Y \subseteq [\gamma_\alpha, \gamma_{\alpha+1})$ for some α; but this is a contradiction since $|[\gamma_\alpha, \gamma_{\alpha+1})| \leq \gamma_{\alpha+1} < \kappa$, and $|Y| = \kappa$. Alternatively, we could have $\operatorname{card}(Y \cap [\gamma_\alpha, \gamma_{\alpha+1})) \leq 1$ for all α. But then $|Y| \leq \lambda < \kappa$, another contradiction.

- κ is a strong limit. Suppose otherwise, that is, there is some $\lambda < \kappa$ with $\kappa \leq 2^\lambda$. Then there exists some injective function $g \xrightarrow{\text{1-1}} {}^\lambda 2$. (Recall that ${}^\lambda 2$ is the set of functions from λ to 2.) Now use g to define a partition $f : [\kappa]^2 \to \lambda$ as follows:

  $$f(\{\alpha, \beta\}) = \text{the least } \gamma \text{ for which } g_\alpha(\gamma) \neq g_\beta(\gamma).$$

  We note that f is total since g is injective.

  However, it is impossible to find a homogenous set of size three under this partition, much less size κ.

∎

Definition 15.7. κ s a *Ramsey cardinal* iff $\kappa \to (\kappa)^{<\omega}_2$.

Lemma 15.8. *If κ is a Ramsey cardinal, then $\kappa \to (\kappa)^{<\omega}_\mu$ for all $\mu < \kappa$.*

Proof. This is also a problem on the final exam. ∎

Lemma 15.9. *Suppose κ is a Ramsey cardinal, and $\langle D, N, E\rangle$ is a directed graph with a binary coloring on the nodes (in particular, D is the set of nodes, $N \subseteq D$ is the set of nodes which are red, and $E \subseteq D \times D$ is the set of edges) such that $|D| = \kappa$ and $|N| = \lambda < \kappa$. Then there is some $\langle D', N', E'\rangle \preceq \langle D, N, E\rangle$ such that $|D'| = \kappa$ and $|N'| = \aleph_0$.*

Proof. Fix a collection of Skolem functions for $\langle D, N, E\rangle$, and let $h(X)$ denote the Skolem hull of X in $\langle D, N, E\rangle$ for $X \subseteq D$.

For every finite $C \subseteq D$, let

$$f(C) = N_C \text{ where } h(C) = \langle D_C, N_C, E_C\rangle .$$

Since $N_C \subseteq N$ for each $C \subseteq D$, we note that $f : [D]^{<\omega} \to \mathcal{P}(N)$, so it is a partition of finite subsets of D into at most 2^λ classes.

Since κ is a Ramsey cardinal, it is weakly compact, and hence strongly inaccessible by Lemma 15.6. Therefore $2^\lambda < \kappa$, and again since κ is a Ramsey cardinal, we conclude by Lemma 15.8 that there is some set $Y \subseteq D$ of cardinality κ for which f is constant on $[Y]^n$ for all $n \in \omega$. For each n, let $X_n = f(C)$ for $|C| = n$.

Note that X_n is countable for all n, since $f(C) = N_C \subseteq D_C$, and the Skolem hull of a finite set is countable.

Now let $\langle D', N', E'\rangle = h(Y)$. We claim that $h(Y) = \bigcup_{C \subseteq_{\text{fin}} Y} h(C)$, where $A \subseteq_{\text{fin}} B$ denotes that A is a finite subset of B: a proof that $y \in h(Y)$ consists of a finite tree with elements of Y at the leaves and Skolem functions at the internal nodes, so for each y we may choose C to be the set containing all the leaves.

$h(Y) \preceq \langle D, N, E\rangle$ by construction; $N' = \bigcup_{n \in \omega} X_n$, which is countable; and taking the Skolem hull preserves cardinality, so $|D'| = |Y| = \kappa$. ∎